

# A Hyperdense Semantic Domain for Discontinuous Behavior in Physical System Models

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**Abstract.** Multiple time models have been proposed for the formalization of hybrid dynamic system behavior. The superdense notion of time is a well-known time model for describing event-based systems where several events can occur simultaneously. Hyperreals provide a domain for defining the semantics of hybrid models that is elegantly aligned with first principles in physics. This paper discusses the value of both time models and shows how approximating different physical effects is best expressed over different domains. Finally, the formalization and interaction of two types of discontinuities observed in hybrid systems, mythical modes and pinnacles, are explored. This analysis helps specify semantics that combine continuous-time behavior with discontinuities in the computational system.

## 1 Introduction

In recent history, the complexity of engineered systems has grown by leaps and bounds, largely because of embedded computation. While embedded systems are well understood and supported by Model-Based Design [14], *Cyber-Physical Systems* (CPS) build on a general paradigm of 'openness' [18] that challenges the current paradigm of system design. This openness manifests, for example, by an application that may execute on different platforms or feature functionality that may be provided by distinctly separate systems. Because it is open, such a CPS cannot rely on integration testing (e.g., [20]) as it is part of the traditional paradigm for embedded system design.

Given the delicate interaction between various component and subsystem behaviors in their implementation, addressing system integration challenges with models is not straightforward. In particular, it is essential to create 'good' models of the physics, that is, models that embody correctly the pertinent physical effects while not giving rise to behaviors that have no physical manifestation. The desiderata for a formalism to model physical systems thus require domain-specific models that inherently reflect the laws of physics. Moreover, the models of the physics must be employable in concert with models of various paradigms such as those for computational and networking functionality in the overall system.

For system-level studies, physics models are generally well described by continuous-time behavior (e.g., based on the foundations of thermodynamics [3, 6]).

At this level, however, physical phenomena that are often part of actuators on the interface with the information technology domain (e.g., electrical switches, hydraulic valves, clutches) typically operate at a time scale too fast to be captured in continuous detail. Instead, such fast behavior is modeled as discontinuous change.

The formalization of behaviors in physical system models builds on a semantic domain that combines evolution in a continuous domain, extended with an integer domain for sequences of mode changes [5, 13]. The resulting  $\mathbb{R} \times \mathbb{N}$  domain, so-called *superdense time* [11], however, is not sufficiently rich to allow a precise mathematical description of the intricate behavior in physical system models around discontinuities.<sup>1</sup> Specifically, an ontology of behavior in hybrid dynamic system models of physical systems developed in previous work [15] includes a class of behaviors called *mythical modes* [16] that maps well onto superdense time. However, another class of behaviors called *pinnacles* requires physical time to advance during discontinuous change, and, therefore, is not amenable to employing superdense time as a semantic domain.

Related work [1, 2, 9] has turned to *hyperreals* from *nonstandard analysis* [10] to define semantics of hybrid dynamic systems. In this paper, the hyperreals are considered as a semantic domain that supports pinnacles. In combination with an integer domain for mythical modes, this leads to a *hyperdense* time domain that supports the various classes of behavior found in hybrid dynamic system models of physical systems. The mathematical formalization is mapped onto a computational implementation that allows for generation of consistent and physically meaningful behavior of interactions between various classes of discontinuities.

Section 2 presents the notion of continuous-time interacting with discontinuities in physical system models. Bond graphs are the formalism to represent these phenomena. Further, pinnacles and mythical modes are introduced in detail and related to the notions of superdense and hyperreal time. Section 3 discusses the interactions among the different modes. Semantics of discontinuous change is explained based on Newton’s cradle modeled as bond graphs. Section 4 concludes.

## 2 Discontinuities in Physical System Models

At a macroscopic level, physical systems are well modeled as continuous-time systems [7]. Continuous phenomena that occur at a time scale much faster than the behavior of interest can be approximated by discontinuities. This section first introduces *bond graphs* [17] as a formalism to model the continuous-time behavior of physical systems. Next, an ideal switching element is added to represent the discontinuity and form *hybrid bond graphs* [12].

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<sup>1</sup> Note that superdense time is typical in a computational approximation of the mathematical representation. However, floating point numbers then represent the continuous domain and the approximation of the continuous domain is in fact not *dense*.

## 2.1 Bond Graphs

Across physics domains (e.g., electrical, hydraulic, thermal, chemical, etc.) thermodynamics identifies two types of variables subject to dynamic behavior representing either: (i) *extensive* quantities or (ii) *intensive* quantities. The dynamics of extensities and intensities are related by *conduction*, that is, when there is a difference in intensities, a change in extensity follows. For example, a difference in velocities between two bodies results in a force acting between them that causes a change in momentum ( $F = m \frac{dv}{dt} = \frac{dp}{dt}$ ). The change of energy, *power*, as the product of the intensity difference, *effort*, and its corresponding change in extensity *flow* then provides a general notion of dynamics across physics domains. For example,  $v \cdot F$  equates power much like in the electrical domain the product of the intensity difference (voltage,  $v$ ) and change of extensity (current  $i$ ) equates power. Consequently, any change in dynamic variable values is the result of an effort,  $e$ , and a flow,  $f$ , acting. Moreover, there are two basic energy-based phenomena: (i) storage of either effort ( $C$ ) or flow ( $I$ ) and (ii) dissipation ( $R$ ). Finally, ideal sources of effort ( $Se$ ) and of flow ( $Sf$ ) define the model context.

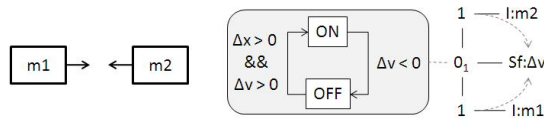
Behavior of the connections then relates the efforts and the flows of all interacting phenomena such that the sum of their product equates 0 (so there is neither dissipation nor storage),  $\sum_i e_i \cdot f_i = 0$ . The two orthogonal implementations of this are that either all efforts are the same while the flows sum to zero, or the converse. In the electrical domain, this corresponds to either Kirchhoff's current law or Kirchhoff's voltage law. In bond graph terminology these connections are represented by *junctions*, the former by a 0 junction ( $\forall_{i \neq j} e_i = e_j$  and  $\sum_i f_i = 0$ ) and the latter by a 1 junction ( $\forall_{i \neq j} f_i = f_j$  and  $\sum_i e_i = 0$ ).

Introducing discontinuities into bond graphs requires an idealized form of discontinuous change in dynamic behavior, which is well represented by a reconfiguration of the junction structure because this structure is ideal. This idealized reconfiguration amounts to a junction between phenomena being active or not [19]. In other words, a 0 junction can be active ( $\forall_{i \neq j} e_i = e_j$  and  $\sum_i f_i = 0$ ) or not ( $\forall_i e_i = 0$ ) and a 1 junction exhibits the dual behavior when active ( $\forall_{i \neq j} f_i = f_j$  and  $\sum_i e_i = 0$ ) or not ( $\forall_i f_i = 0$ ). Note that when a junction is not active, indeed no power flows across it. These junctions that can change their mode from active (*on*) to inactive (*off*) are called *controlled junctions*.

## 2.2 The Logic of Discontinuities in Physics Models

A controlled junction is equipped with a finite state machine (FSM) that determines the junction *on* or *off* mode, which involves capturing: (i) how the state of the FSM maps onto the *on* and *off* mode of the junction and (ii) how the physical quantities map onto transition conditions of the FSM. Continuity of power implies that discontinuities in physical quantities result from a lack of detail in modeled phenomena, which come in two classes: (i) storage and (ii) dissipation. The discontinuous behavior that emerges in turn for each of these is discussed next.

**Pinnacles** Multibody collisions are often modeled by discontinuous velocity changes. In a hybrid bond graph model, a collision between two bodies,  $m_1$  and  $m_2$ , can be modeled as depicted in Fig. 1. The two bodies are modeled as inertias,  $I$ , connected to a common velocity,  $1$ , junction. These junctions represent the respective velocities,  $v_1$  and  $v_2$ , which are connected via a common force,  $0$ , junction. This  $0$  junction is controlled and when *off* it exerts force  $0$  on both bodies. Upon collision, the  $0$  junction turns *on* and it now enforces a velocity balance such that  $v_1 - v_2 + \Delta v = 0$ , where  $\Delta v$  is computed by an ideal flow source,  $Sf$ , as  $\Delta v = v_1^- - v_2^-$ , with the ‘ $-$ ’ superscript referring to signals immediately preceding the collision.



**Fig. 1.** Ideally plastic collision

The FSM controlling the *on/off* mode of the  $0$  junction switches from *off* to *on* when the bodies make contact ( $\Delta x > 0$ ) and when they are moving toward one another ( $\Delta v > 0$ ). Here the  $\Delta v > 0$  is essential to model that there is a collision as opposed to the bodies only being in contact. As soon as the bodies move away from one another ( $\Delta v < 0$ ), the  $0$  junction switches to *off*, irrespective of whether the bodies are touching.

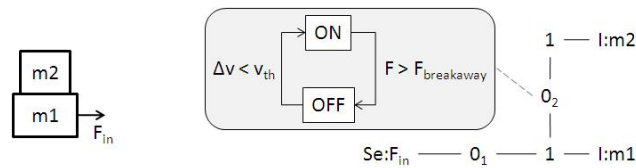
During behavior generation, when  $\Delta x > 0 \ \&\& \ \Delta v > 0$  holds, a collision occurs and the flow source enforcing the velocity difference  $\Delta v^-$  becomes active. Based on this velocity difference and conservation of momentum ( $\sum_i m_i v_i^- = \sum_i m_i v_i$ ), the velocities upon collision can be computed. The state of the velocity of the bodies is then reinitialized and this leads to the condition  $\Delta v < 0$  being satisfied. Thus, a consecutive mode change occurs where the FSM moves to the *off* mode again. In the *off* mode the bodies behave as independent masses, and, therefore, no further changes in the physical state occur. Since the discrete mode changes have thus converged, the system proceeds to evolve in continuous time.

The end result is that the bodies  $m_1$  and  $m_2$  evolve according to a mode of continuous evolution. With a point in time at which two mode changes occur: (i) first, a collision mode occurs that necessitates a reinitialization (discontinuous change) and (ii) second, the system changes back to a mode of continuous evolution. The collision mode that is active only as a reinitialization of physical state is referred to as a *pinnacle* [13].

**Mythical Mode Change** Now, consider two bodies  $m_1$  and  $m_2$  with  $m_2$  at rest on top of  $m_1$ . When at a point in time a large enough external force is exerted on  $m_1$ ,  $m_1$  will start moving with a corresponding velocity. However, if the force

is sufficiently large that the breakaway friction force  $F_{breakaway}$  between  $m_1$  and  $m_2$  is exceeded,  $m_2$  may remain at rest.

A hybrid bond graph model of such a system is depicted in Fig. 2. An ideal source of effort exerts a force on  $m_1$  because connected to the common velocity 1 junction that represents the velocity of  $m_1$ . When *on*, a controlled 0 junction connects the 1 junction that represents the velocity of  $m_2$ , which forces  $m_1$  and  $m_2$  to move with the same velocity. The FSM for the controlled 0 junction shows that the junction changes to its *off* mode when the force between  $m_1$  and  $m_2$  exceeds the breakaway force,  $F > F_{breakaway}$ . In the *off* mode, the 0 junction exerts 0 force on both  $m_1$  and  $m_2$ , and so they move independently. The FSM also shows that if the velocity difference between  $m_1$  and  $m_2$  falls below a threshold velocity ( $\Delta v < v_{th}$ ) the two bodies ‘stick’ to each other again.



**Fig. 2.** Two bodies with a breakaway force

During behavior generation, initially the 0 junction is in its *on* mode because the bodies are at rest with one atop the other and the system evolves in continuous time. Now, at the point in time where  $F_{in}$  changes discontinuously new velocities for both  $m_1$  and  $m_2$  are computed. These velocities, however, may require a force to be exerted on  $m_2$  that causes the condition  $F > F_{breakaway}$  to be satisfied and the 0 junction changes to its *off* mode. In the *off* mode, if the velocity difference is sufficiently large, no further mode changes occur and the system proceeds to evolve in continuous time.

At the point in time at which a discontinuous force is exerted the corresponding velocities and forces are computed and based on the newly computed values the connection between the two bodies changes mode such that they are dynamically independent. Since there is no effect of the external force on the velocity of  $m_2$ , in order to arrive at the proper values for reinitialization of  $v_1$  and  $v_2$  the mode where the external force becomes active while  $m_1$  and  $m_2$  are still connected is considered to have no effect on the physical state, which is referred to as a *mythical mode* [13].

### 2.3 Introduction to Superdense Time

Time-event sequence is a semantic domain for describing event-based models. Intuitively, time-event sequences are instantaneous events separated by non-negative real numbers that describe time durations between the events. Events

separated by zero duration are simultaneous, but have a well-defined causal ordering.

Superdense time was introduced to represent time-event sequences as functions of time [11]. Superdense time is a totally ordered subset of  $\mathbb{R}_+ \times \mathbb{N}$ , where the non-negative real number represents the real time and the natural number represents the causal ordering. Simultaneous events at time  $t$  are mapped to  $(t, 0), (t, 1), \dots$  superdense time instants such that the ordering of the events is preserved.

The (total) ordering of superdense time is given by the following definitions:  $(t, n) = (t', n') \Leftrightarrow t = t' \wedge n = n'$ , and  $(t, n) < (t', n') \Leftrightarrow t < t' \vee (t = t' \wedge n < n')$ . Therefore, superdense time is a time model that can be used to describe simultaneous events as functions of time, while retaining the causality of events.

Mythical modes emerge as an artifact of logical inference to determine a new mode in which physical state can change. As such, mythical modes do not affect the dynamic state of a physical system. Moreover, different logic formulations may traverse different mythical modes yet still arrive at the same resulting mode where physical state changes can occur. Consequently, the logical evaluation has no corresponding manifestation in the dynamic state of a physical system and occurs at a single point in time along a logical inferencing dimension. This behavior corresponds to the superdense semantic domain.

## 2.4 Introduction to Hyperreal Time

Calculus comprises two different approaches to capturing infinitely small values: either through the use of limits, or by the extension of the field of reals with infinitesimals. An infinitesimal  $\epsilon$  is any number, such that  $|\epsilon| < \frac{1}{n}$ , for any  $n \in \mathbb{N}$ .

Intuitively, the idea behind hyperreals is to extend the dense field of  $\mathbb{R}$  with infinitely many points around each real number such that any real sentence that holds for one or more real functions also holds for the hyperreal natural extensions of these functions [10] (transfer principle).

In the ultrapower construction [8], hyperreals are represented as sequences of real numbers  $u_1, u_2, \dots, u_n \in \mathbb{R}^n$  with real numbers embedded as constant sequences (i.e., a real number  $r$  is the sequence of  $r, r, \dots, r \in \mathbb{R}^n$ ). These sequences, together with elementwise addition and multiplication operations, form a commutative ring but not a field (since the multiplication of two non-zero numbers could result in zero:  $0, 1, 0, \dots \times 1, 0, 1, \dots = 0, 0, \dots$ ). This issue is remedied by considering equivalence classes of  $\mathbb{R}^n$  defined by a free ultrafilter  $U$  of  $\mathbb{N}$ .

Let  $J$  be a non-empty set. An ultrafilter on  $J$  is a nonempty collection  $U$  of subsets of  $J$  having the following properties:  $\emptyset \notin U$ ;  $A \in U$  and  $B \in U$  implies  $A \cap B \in U$ ;  $A \in U$  and  $A \subseteq B \subseteq J$  implies  $B \in U$ ; for all  $A \subseteq J$ , either  $A \in U$  or  $J \setminus A \in U$ . For any  $x \in J$  there is a principal ultrafilter  $\{A \subseteq J \mid x \in A\}$ . Finally, any non-principal ultrafilter is called a free ultrafilter.

Given an ultrafilter  $U$ , an equivalence relation  $=_U$  can be defined over  $\mathbb{R}^n$ :  $u =_U v$  holds for sequences  $u = u_1, \dots, u_n$  and  $v = v_1, \dots, v_n$  if and only if  $\{i \mid u_i < v_i\} \in U$ . The hyperreals are then defined as the quotient of  $\mathbb{R}^n$  by  $U$ ,  ${}^*\mathbb{R} = \mathbb{R}^n / U$ . Now,  ${}^*\mathbb{R}$  is an ordered field for which the transfer principle holds.

As a semantic domain, hyperreals have the advantage that between any *real* time instant there are many ordered time instants. Such extension of time greatly simplifies the semantic specification of discontinuities, in particular, the description of pinnacles that represent fast physical behaviors where the dynamic state changes discontinuously. As a result, a pinnacle corresponds to a distinct state of physical behavior. In physics, such a distinct state corresponds to a distinct point in time. Because the continuous behavior represented by a pinnacle is considered to occur infinitely fast, time is considered to advance by an infinitesimal amount for a pinnacle to implement the physical state change. This behavior corresponds to the hyperreal domain.

It is a straightforward extension to introduce a hyperdense time model as a “combination” of the super-dense and hyperreal time models. We define the hyperdense time  ${}^*\mathbb{R}_+ \times \mathbb{N}$  as the product of the non-negative hyperreals and natural numbers. Such a time model can be used for representing both infinitesimal time advancements, as well as establishing a causal ordering at any hyperreal time instant.

### 3 Semantics of Discontinuity Behavior

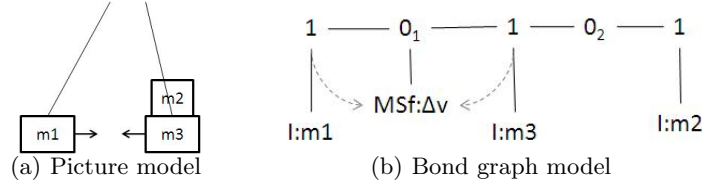
The formalized models of time provide the ingredients for a semantic domain that is sufficiently rich to formalize the pinnacle and mythical mode behavior at discontinuities as well as combinations.

#### 3.1 Interacting Pinnacles and Mythical Modes

With superdense time as a semantic domain for mythical modes and hyperreals for pinnacles, models that engender both build on a combined hyperdense semantic domain. The particular value of such a precise semantic description lies in the ability to develop consistent computational behavior generation algorithms. Because of the discreteness of computational values, the semantic domain of values in computational models can represent neither superdense nor hyperreal domains. Therefore, the behavior generation algorithms must include sophistication that addresses the differences between superdense and hyperreal semantic domains. The computational implementation of each of the semantic domains and their interaction is described based on an illustrative example.

In Fig. 3(a), a variant of Newton’s Cradle is shown. One of the bodies,  $m_3$ , has another body,  $m_2$ , positioned on top of it. Stiction effects between  $m_2$  and  $m_3$  cause them to behave as one body with combined mass as long as the breakaway force between them,  $F_{breakaway}$ , is not exceeded. A body,  $m_1$ , may collide with  $m_3$  according to a perfectly elastic collision,  $\Delta v_{32} = -\epsilon \Delta v_{32}^-$ , where  $\Delta v$  is the difference in velocities ( $v_3 - v_2$ ) after the collision and  $\Delta v_{32}^-$  is the difference in velocities before the collision.

The bond graph model in Fig. 3(b) shows the three masses each connected to a common velocity junction, 1, with the velocity of the directly connected mass on all ports. Common force junctions, 0, connect the 1 junctions and are



**Fig. 3.** Newton's Cradle for advanced maneuvers

*controlled junctions* such that a finite state machine determines their *on* or *off* state. A modulated flow source  $MSf$  models the collision with restitution  $\epsilon = 0.8$ . If the controlled junction  $0_1$  is in its *on* state, this flow source enforces a difference in velocities of  $m_1$  and  $m_3$ , possibly accounting for the rigidly connected mass  $m_2$ . If the controlled junction  $0_1$  is in its *off* state, a 0 force is exerted on both  $m_1$  and  $m_3$  (possibly accounting for  $m_2$ ). In its *off* state, the controlled junction  $0_2$  exerts a 0 force as well, which is when the force at the contact point between  $m_2$  and  $m_3$  exceeds the breakaway force. Note that for the case when  $m_2$  and  $m_3$  move independently in continuous time no viscous friction is modeled for clarity purposes. If the difference in velocities between  $m_2$  and  $m_3$  falls below a threshold level, the stiction effect becomes active, modeled by  $0_2$  changing to its *on* state. In the *on* state, a difference in velocities of  $m_2$  and  $m_3$  of 0 is enforced.

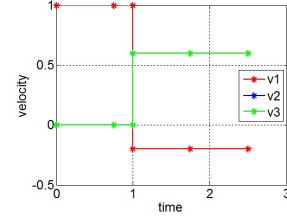
Upon collision of  $m_1$  and  $m_3$ , if the difference in velocities  $v_2$  and  $v_3$ ,  $\Delta v_{23}$ , is less than the threshold velocity  $v_{th}$ , stiction is active and  $m_2$  and  $m_3$  behave as one body with mass  $m_2 + m_3$ . When  $m_2 + m_3 > m_1$ ,  $m_1$  will have a return velocity and start moving in the opposite direction compared to the velocity before the collision. However, the momentum of  $m_1$  may be such that an impulsive force [4] arises between  $m_2$  and  $m_3$  that triggers the  $F_{breakaway}$  transition, causing the two bodies to move independently. In this case, if  $m_1 = m_3$ , there is no return velocity of  $m_1$  but instead it acts as in the case of Newton's Cradle where  $m_3$  assumes all of the momentum of  $m_1$  while  $m_1$  comes to rest. In this case the velocity of  $m_2$  is not affected by the collision.

The importance of semantic domain that combines both superdense as well as hyperreals is clearly illustrated by this example. While the condition for  $0_2$  to switch from *on* to *off* occurs in 0 time, the condition for  $0_1$  to switch from *on* to *off* occurs in infinitesimal,  $\epsilon$ , time. A critical consequence of this phenomenon is that, although in reasoning about the system,  $0_1$  first changes its state to *on*, after which the change of state in  $0_2$  to *off* is determined, the change of state in  $0_1$  back to *off* is not effected until *after* the change of state in  $0_2$  to *off*.

In Table 1 and in Table 2,  $0_{F13}$  and  $0_{F23}$  are the junctions at a select force impact, while  $p_{m1}$ ,  $p_{m2}$ , and  $p_{m3}$  are the momenta for each of the masses. The sequences of mode changes are depicted in Table 2, which shows clearly the difference in effects as the model evolves in superdense and in hyperreal time. The ability to differentiate between  $t = \langle t_{collide}, 1 \rangle$  and  $t = \langle t_{collide} + \epsilon, 0 \rangle$  makes it possible to distinguish the pinnacle effect of  $0_1$  from the mythical mode effect of  $0_2$ . Otherwise,  $0_1$  would have switched back *off* simultaneously with  $0_2$  switching

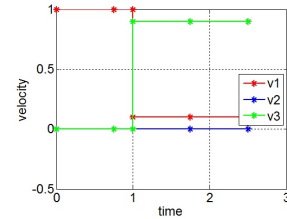


time	$0_{F13}$	$0_{F23}$	$p_{m1}$	$p_{m2}$	$p_{m3}$
$t = \langle t_{collide} - \epsilon, 0 \rangle$	<i>off</i>	<i>on</i>	1	0	0
$t = \langle t_{collide}, 0 \rangle$	<i>on</i>	<i>on</i>	-0.2	0.6	0.6
$t = \langle t_{collide} + \epsilon, 0 \rangle$	<i>off</i>	<i>on</i>	-0.2	0.6	0.6



**Table 1.** Mode changes for  $v_{th}^2 = 0.1$  and  $F_{th} = 0.95$

time	$0_{F13}$	$0_{F23}$	$p_{m1}$	$p_{m2}$	$p_{m3}$
$t = \langle t_{collide} - \epsilon, 0 \rangle$	<i>off</i>	<i>on</i>	1	0	0
$t = \langle t_{collide}, 0 \rangle$	<i>on</i>	<i>on</i>	-0.2	0.6	0.6
$t = \langle t_{collide}, 1 \rangle$	<i>on</i>	<i>off</i>	0.1	0.9	0.0
$t = \langle t_{collide} + \epsilon, 0 \rangle$	<i>off</i>	<i>off</i>	0.1	0.9	0.0



**Table 2.** Mode changes for  $v_{th}^2 = 0.1$  and  $F_{th} = 0.5$

*off.* This would either: (i) not allow modeling of inferencing (mythical) modes or (ii) have the collision effect (incorrectly) computed for  $m_2$  and  $m_3$  comprising a combined mass of  $m_2 + m_3$ .

## 4 Conclusions

Superdense and hyperreal time notions provide a comprehensive basis for formalizing a computational semantics. The interplay between them enables the design of models that include continuous-time behavior interacting with discontinuities of physical system models. Such formalization is of great value, in particular in simulation technologies, because of the benefits in a consistent projection of behavior according to the laws of physics into a computational representation.

Based on the bond graph modeling formalism, a formalization is developed for combining continuous-time with discontinuities. The combination provides a theoretical reference for the computational integration of different effects of discontinuities observed across multiple domains. Moreover, the work allows for a consistent mapping onto corresponding algorithms that lack hyperreals as execution domain. Most prominently, the research addresses how to combine behavior because of logical switching with physics-based switching behavior.

Formalization in this domain often foregoes the collision mode by reinitializing velocities as a transition action in a state machine. Though there is no fundamental difference in behavior generation, such a representation makes it exceedingly complicated to attribute a sound theory of physics to discontinuous behavior. Instead, this paper relates pinnacles and mythical modes to different notions of time so as to formalize their interaction in a computational sense.

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