Graph Grammars

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February 18, 2004

Abstract

This presentation introduces graph transformations.

Structure of the talk

- Graph Theory
- Subgraph isomorphism problem and algorithms
- Graph Rewriting: General Framework & Difficulties
- Double & Single pushout approaches

Graph Theory

- A *(labelled, directed) graph* $G = (V, E, source, target, label)$ consists of a finite nonempty set V (vertices) and a finite set $E \subseteq V \times V$ (edges), along with two mappings *source* and *target* assigning a source and a target node to each edge, and a mapping *label* assigning a *labelling* symbol from a given alphabet to each node and each edge.
  - order: $|V|$, size: $|E|$
  - An arc $e = (v, w)$ is said to be incident with vertices v (source) and w (target).
  - $G.inc(v)$, $G.outc(v)$, are two mapping assigning a given vertice v to its in and out connections,respectively.

- Subgraph, induced subgraph. Given two graphs, $G = (V, E)$, $G' =$

(V', E'), G' is said to be a subgraph of G if $E' \subseteq E$. The subgraph of G induced by G' is the graph $(V', E \cap V' \times V')$.

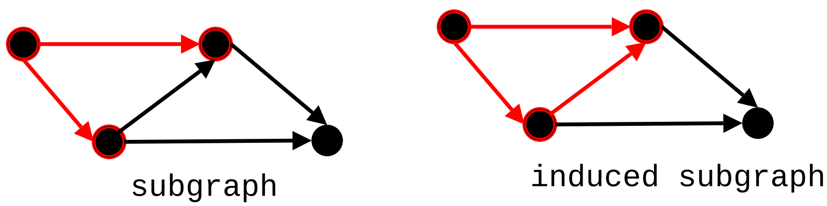


Figure 1: Sugraph / Induced subgraph (labels omitted)

- *Homomorphism (mapping)*. Given two graphs $G = (V, E)$, $G' = (V', E')$, a vertex mapping $H : V \to V'$ is said to be an homomorphism from G to G' if it is edge preserving: Given two vertices $u, v \in V$, $H(u)$ is adjacent to $H(v)$ whenever u is adjacent to v.

- *Graph Isomorphism*. A bijective (one-to-one, surjective) homomorphism.

- *(Partial) Subgraph Isomorphism.* An injective (one-to-one) homomorphism. Also called *monomorphism* in the literature.

- *Induced (Complete) Subgraph Isomorphism.* A injective homomorphism reflecting in G the structure of the induced subgraph of G'.

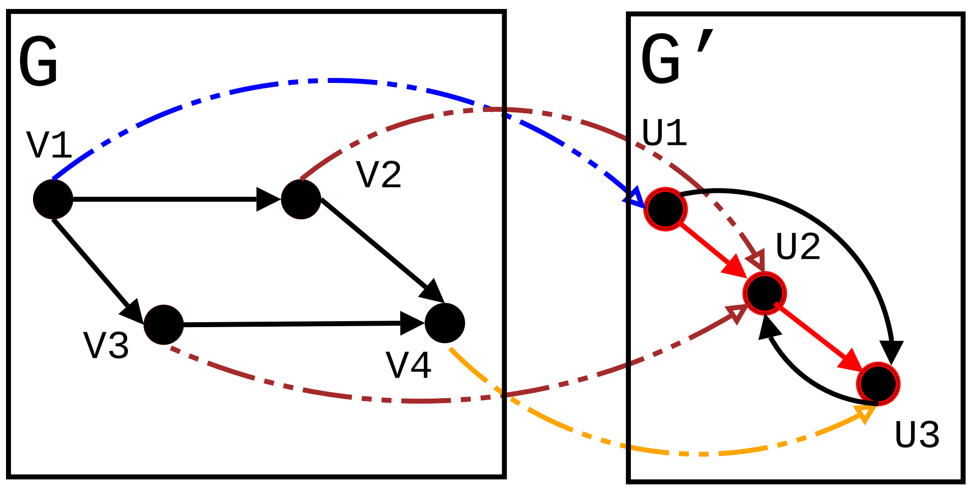


Figure 2: Homomorphism

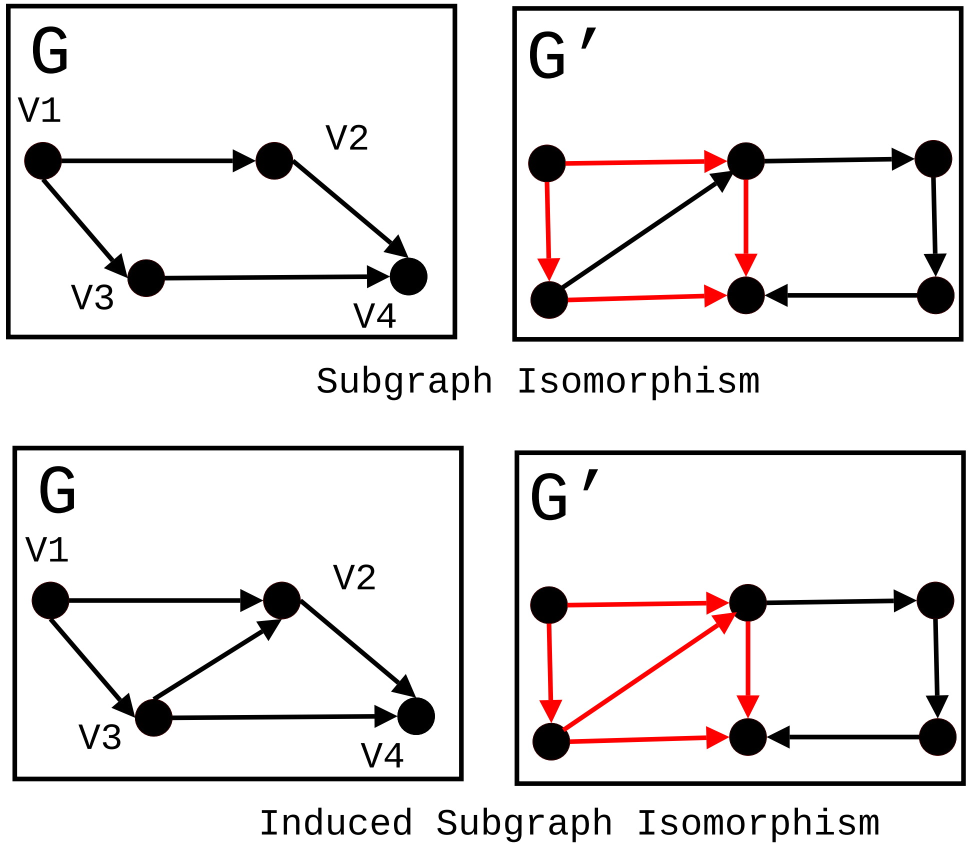


Figure 3: Complete,Partial subgraph isomorphisms

Subgraph Isomorphism Problem & Algorithms

- The first step to rewrite a host graph is to match the left-hand side of a rule.

- Subgraph Isomorphism is NP-Complete for general graphs.

- In general, the algorithms are based on the idea of *backtracking*: Extend a *partial* solution, one variable at a time, until a *complete* solution is reached or the partial solution cannot be extended anymore; this can be viewed by a *backtracking tree*.

- Most algorithms try to limit the time-space explosion by pruning the backtracking tree.

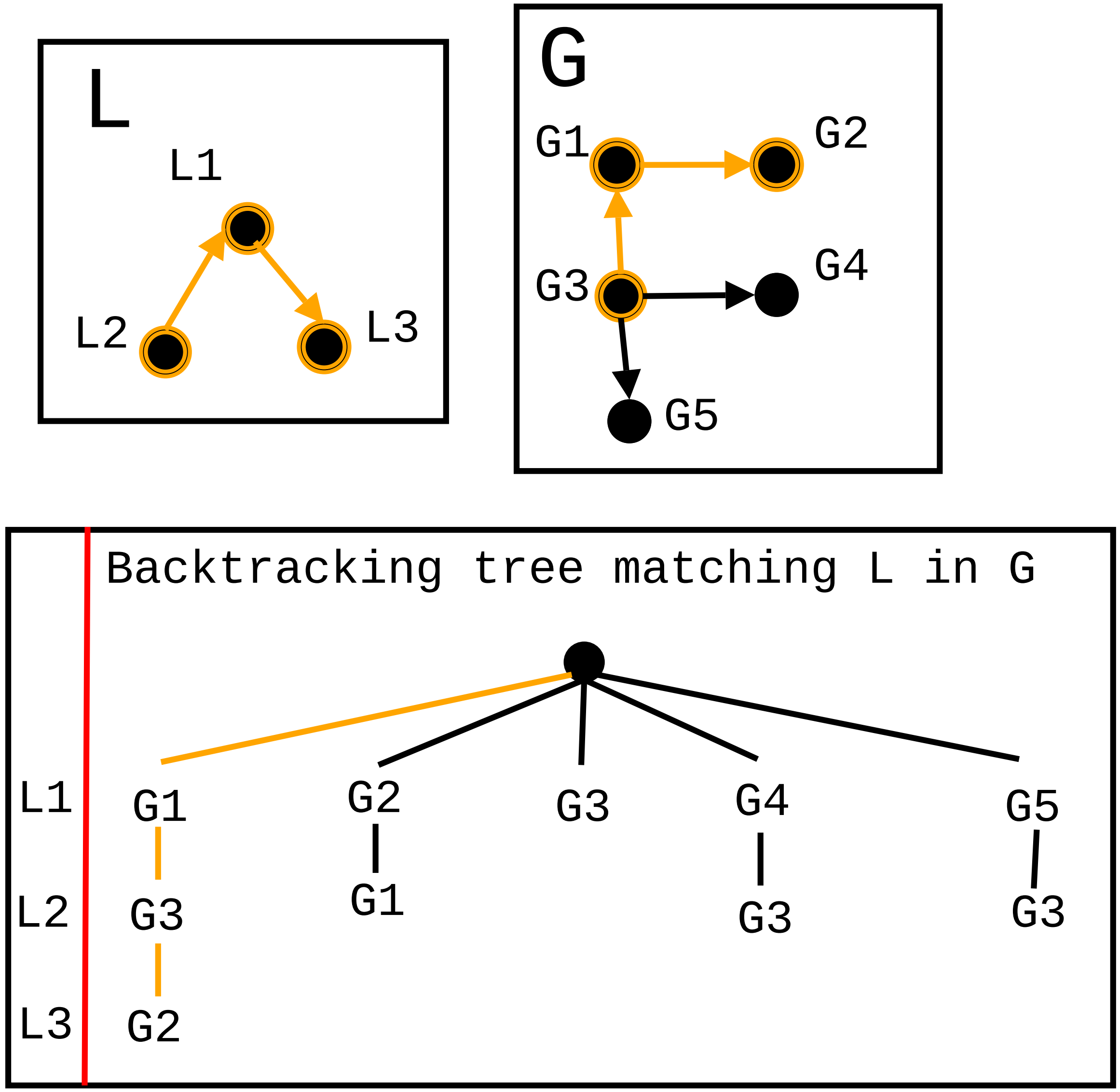


Figure 4: Execution of a typical subgraph isomorphism algorithm

- Pruning involves testing that the current partial solution cannot evolve toward a complete solution. Given a graph L to be matched into a graph G, suppose we have the following partial solution M=$((l_1, g_1), (l_2, g_2), ..., (l_i, g_i))$. Now, we are extending M with (l_{i+1}, g_{i+1}), and we want to test if this path can possibly lead to a complete solution. Possible pruning:
  - If $L.outc(l_{i+1}) > G.outc(g_{i+1})$: prune the tree!
  - If $L.inc(l_{i+1}) > G.inc(g_{i+1})$: prune the tree!
  - *Ullmann's algorithm* [4]: Check that, for all vertices l_v adjacent to l_{i+1}, there is at least one vertex g_v adjacent to g_{i+1} such that $M \cup \{(l_{i+1}, g_{i+1}), (l_v, g_v)\}$ is a valid partial solution. If not, prune the tree!

- Other approaches to subgraph isomorphism:
  - Is it possible to solve the problem in polynomial time if some

preprocessing is used? A very interesting paper [3] shows how to solve the problem by means of decision trees:

* A decision tree is constructed from a database of *model* graphs. (By generating the set of all permutations of the adjacency matrix of all graphs, and organizing it into a decision tree)
* At run-time, easily determine if there is a subgraph isomorphism between an unknown *input* graph and some of the model graphs.
* Run in time $O(n^2)$ if preprocessing is neglected
* But, the size of the decision tree contains an exponential number of nodes...
* Useful if the model graphs are small, and real time behavior is needed.

Graph Rewriting: General Idea [2]

- Basic Idea: Iteratively apply rules to transform a *host* graph G.

- General framework: a (graph transformation) *rule* r=(L,R,K,glue,emb,appl) consists of:

  - Two graphs: a left-hand side L and a right-hand side R
  - A subgraph K of L; the *interface* graph.
  - A homomorphism $glue$, relating K to the right-hand side R
  - An embedding relation emb, relating nodes of L to nodes of R.
  - A set $appl$ specifying the applications conditions for the rule

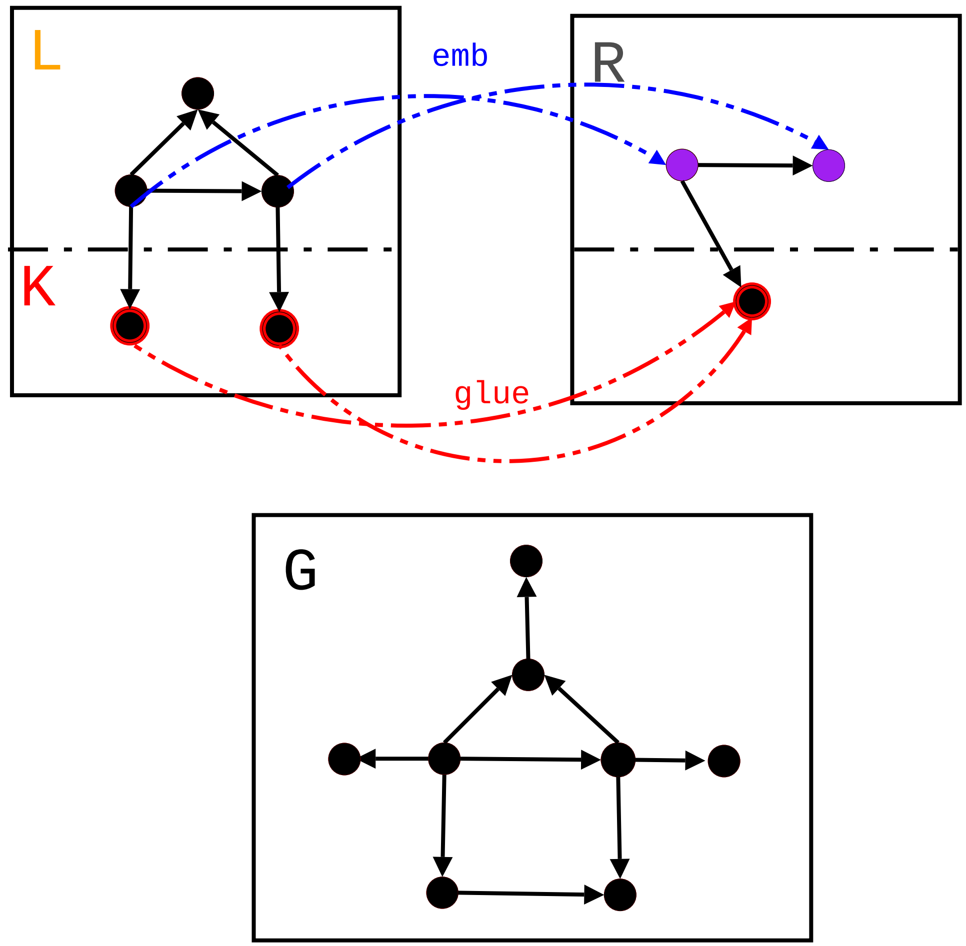
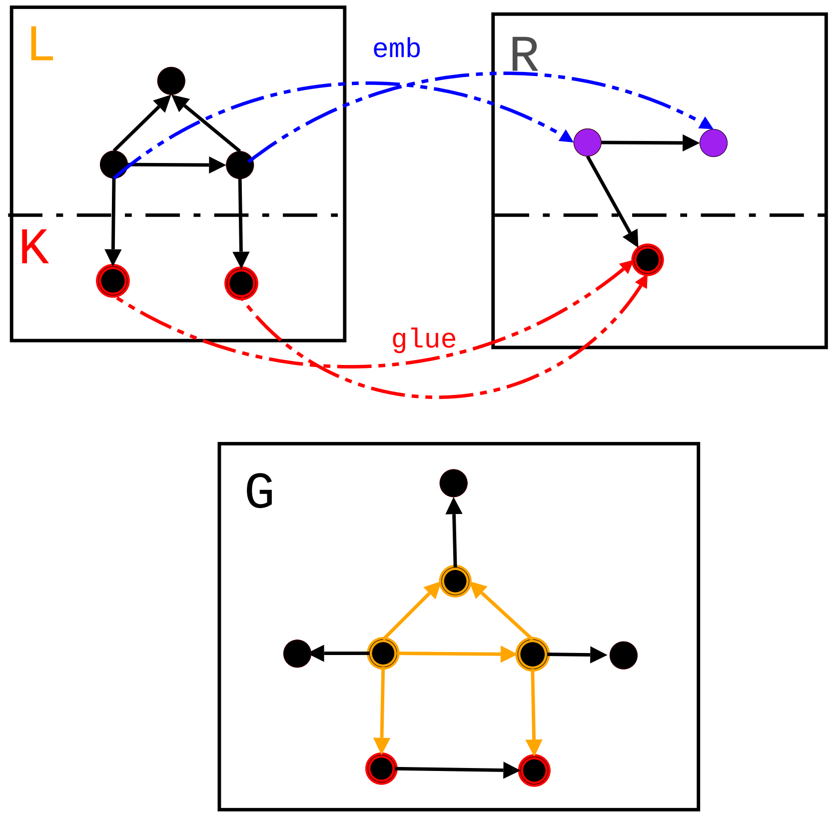


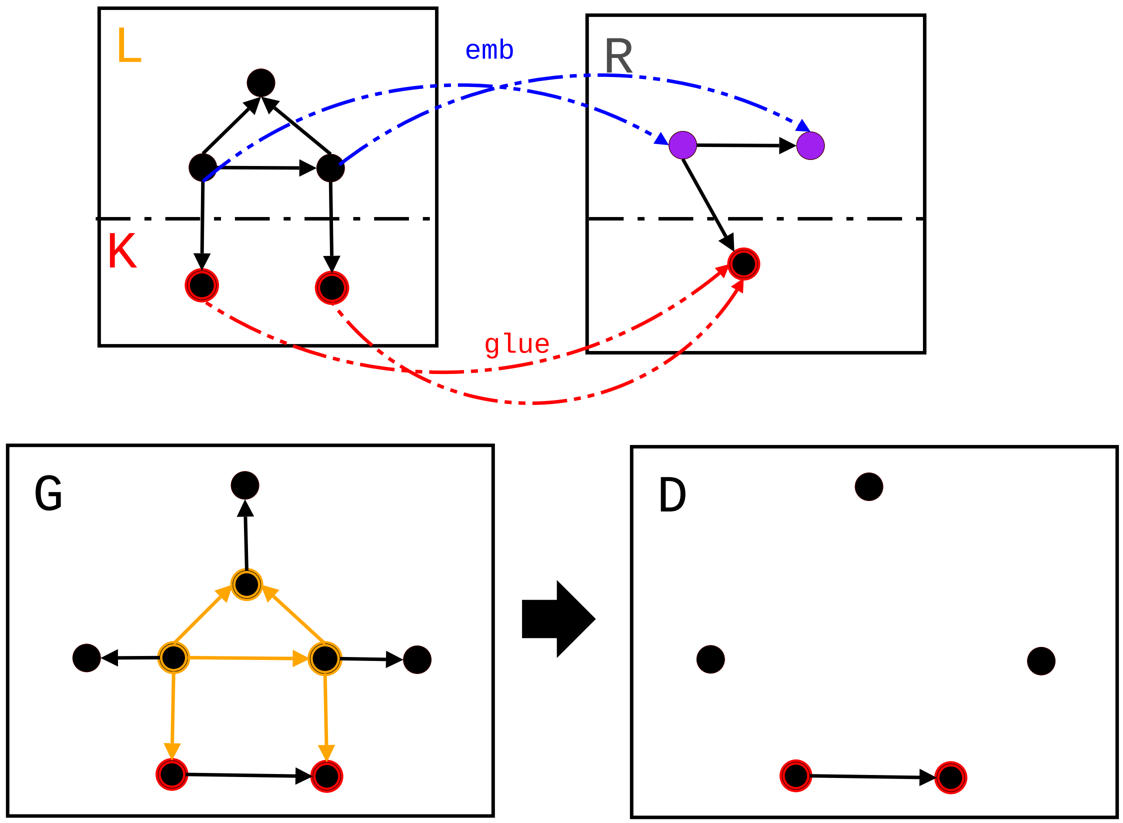
Figure 5: Example of a graph transformation rule

- Step 1: Match L in the host graph G:

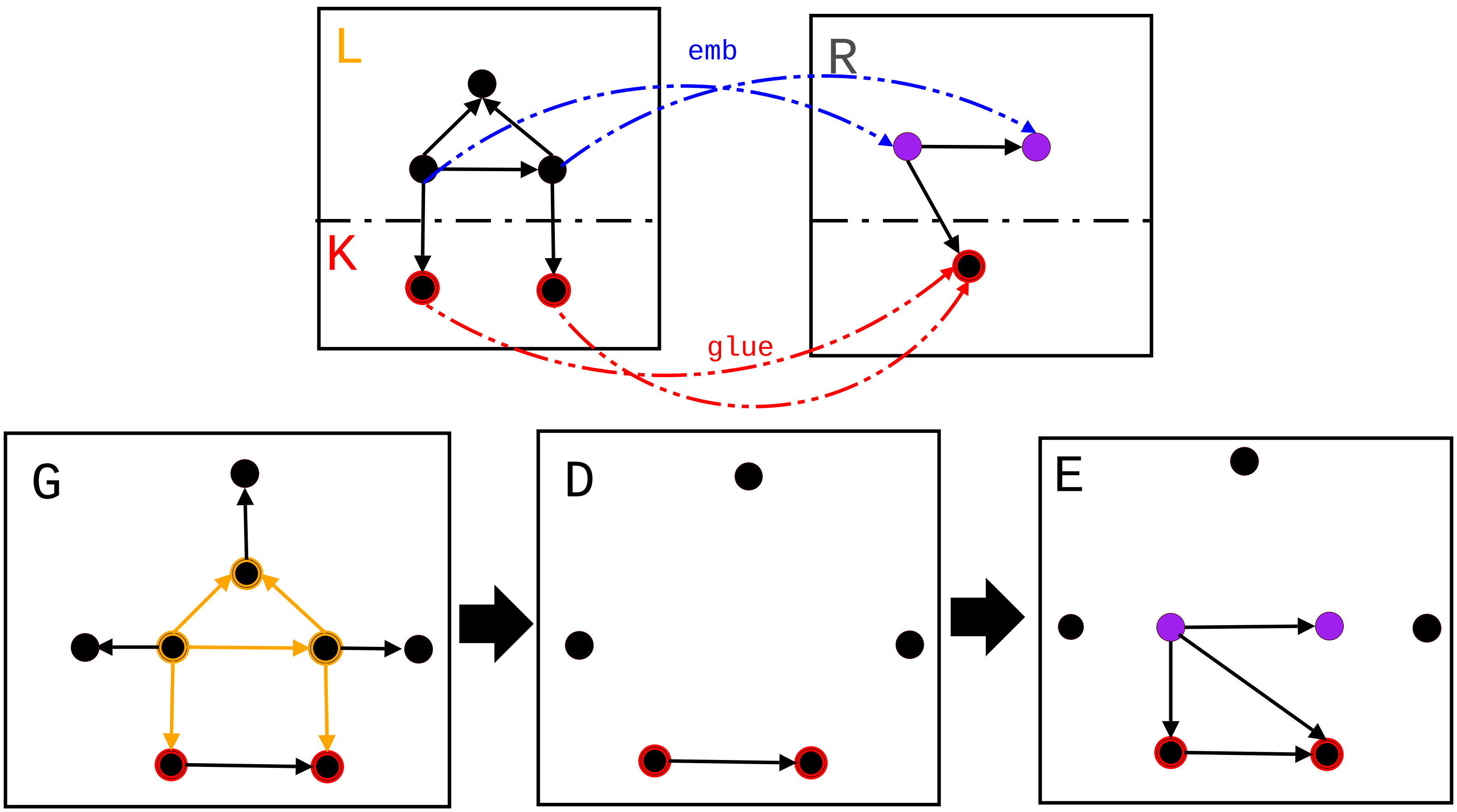


- Step 2: Check the applications conditions (Assume none in this example).

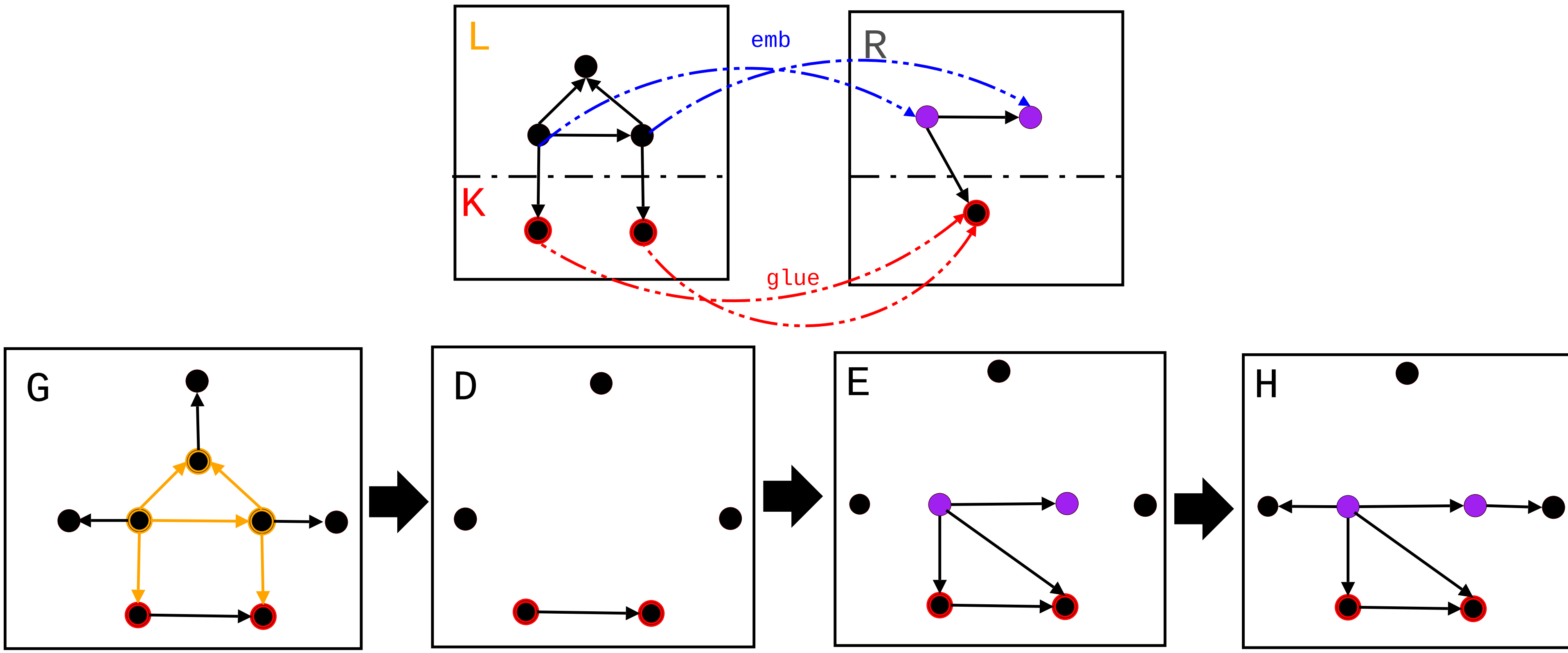
- Step 3: Remove from G the part of the isomorphic match that correspond to L (i.e. keep the interface subgraph K), along with all dangling edges. This yields the *context* graph D. In short, $D = G - (L - K)$:



- Step 4: Glue the context graph D and the right-hand side R, according to the *glue* homomorphism. This yields the gluing graph E:



- Step 5: Embed the right-hand side R into D, according to the relation *emb*. This yields the derived graph H:

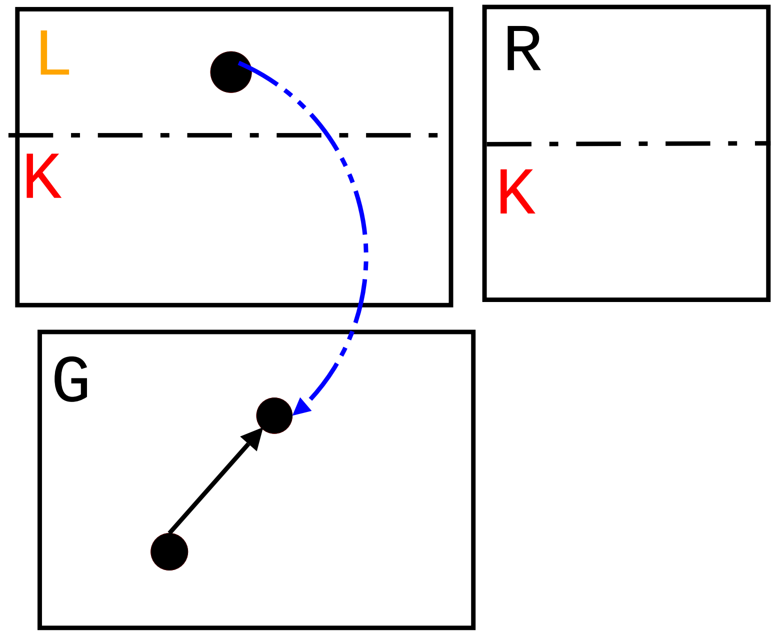


- Several Difficulties occur with graph grammars:
  - Do we require an isomorphism between the LHS and the host graph?
  - Do we delete dangling edges when replacing the LHS by the RHS? Or we do not allow such rule to execute?
  - How do we organize the rules?

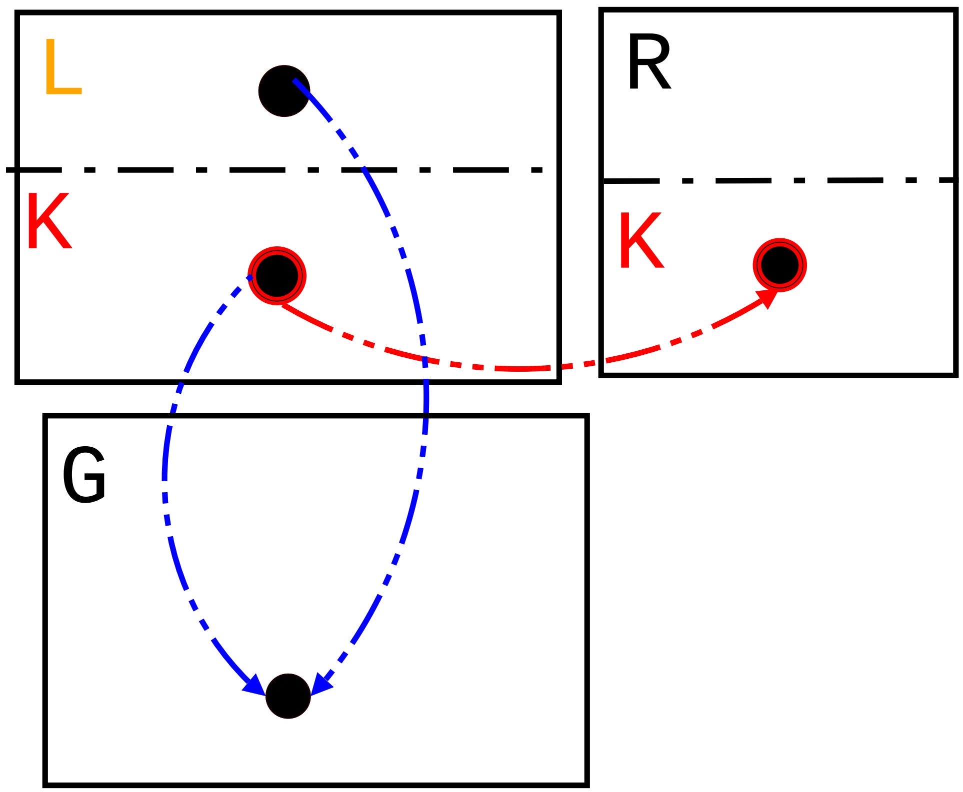
Double pushout approach (DPO) [1]

- The first *algebraic* approach
- *Algebraic*: Basic algebraic constructions (from *category* theory) were used to define the transformation.
  - In DPO: Two pushouts (glues) are performed in the transformation. The first pushout correspond to the deletion of the LHS in the host graph. The second pushout correspond to the insertion of the RHS.
- No embedding function.
- The homomorphisms between K and L; K and R must be injective.

- Does *not* require an isomorphism between L and the host graph G. Instead, a less restrictive approach is used. Two conditions are added for the application of a rule. (also known as the gluing conditions):
  - *Dangling condition*: If a vertex is deleted, then the production *must* specify the deletion of the edges incident to that node.G



– *Identification condition*: Every element that should be deleted in G has only one pre-image in L



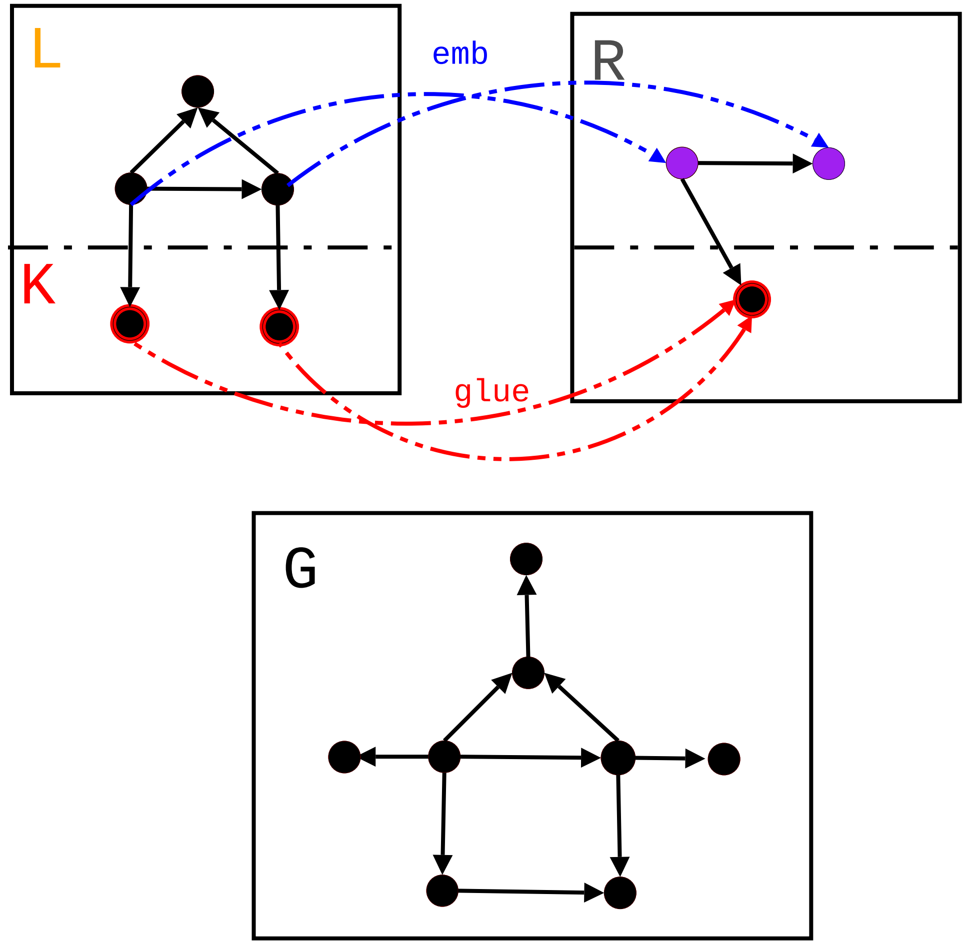


Figure 6: General graph transformation rule

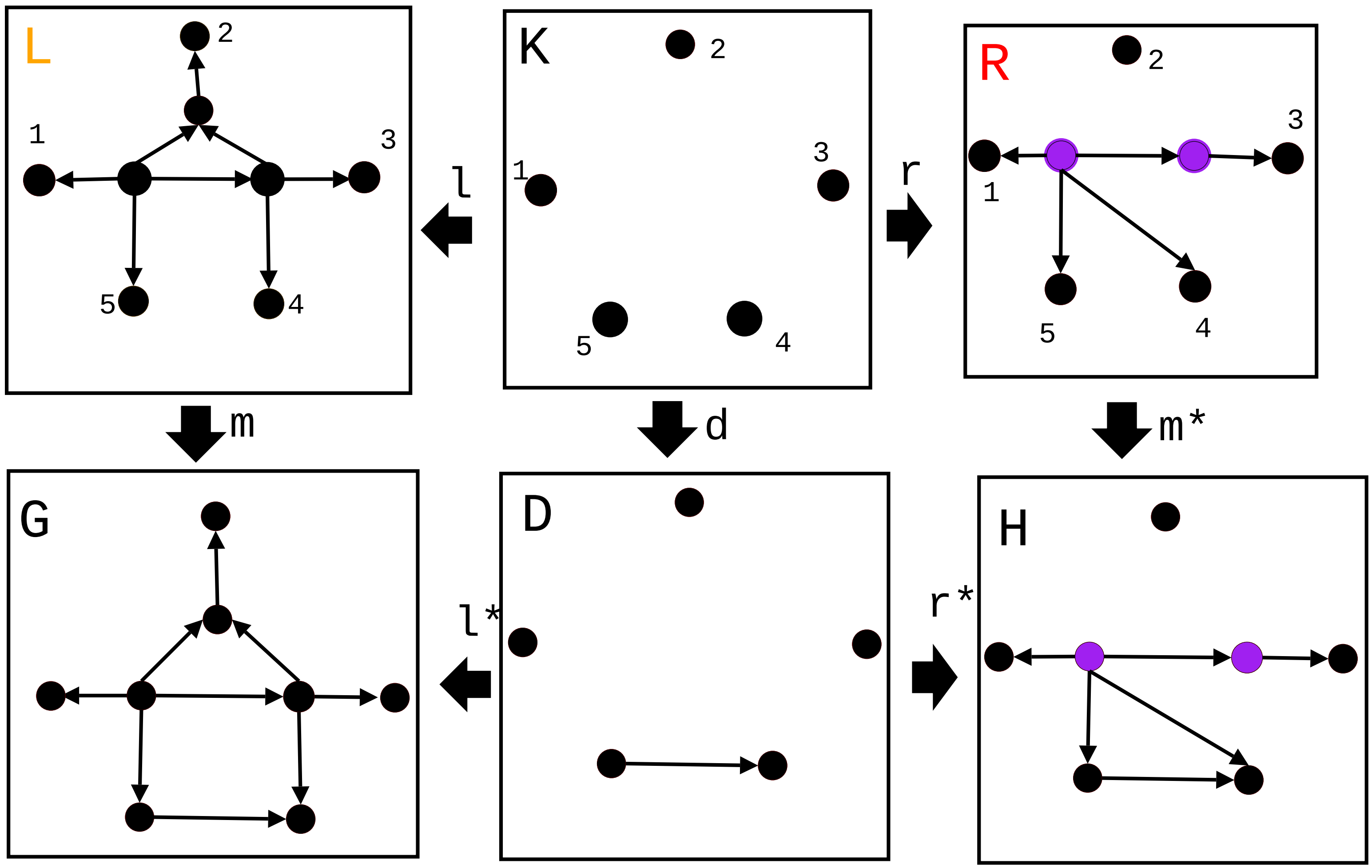


Figure 7: DPO graph transformation rule

Single pushout approach (SPO) [1]

- The SPO approach was introduced to add expressiveness to the derivations.
- No gluing conditions.
- No embedding function.
- Deletion has priority over preservation
- The interface graph is expressed as a partial homomorphism between L and R.

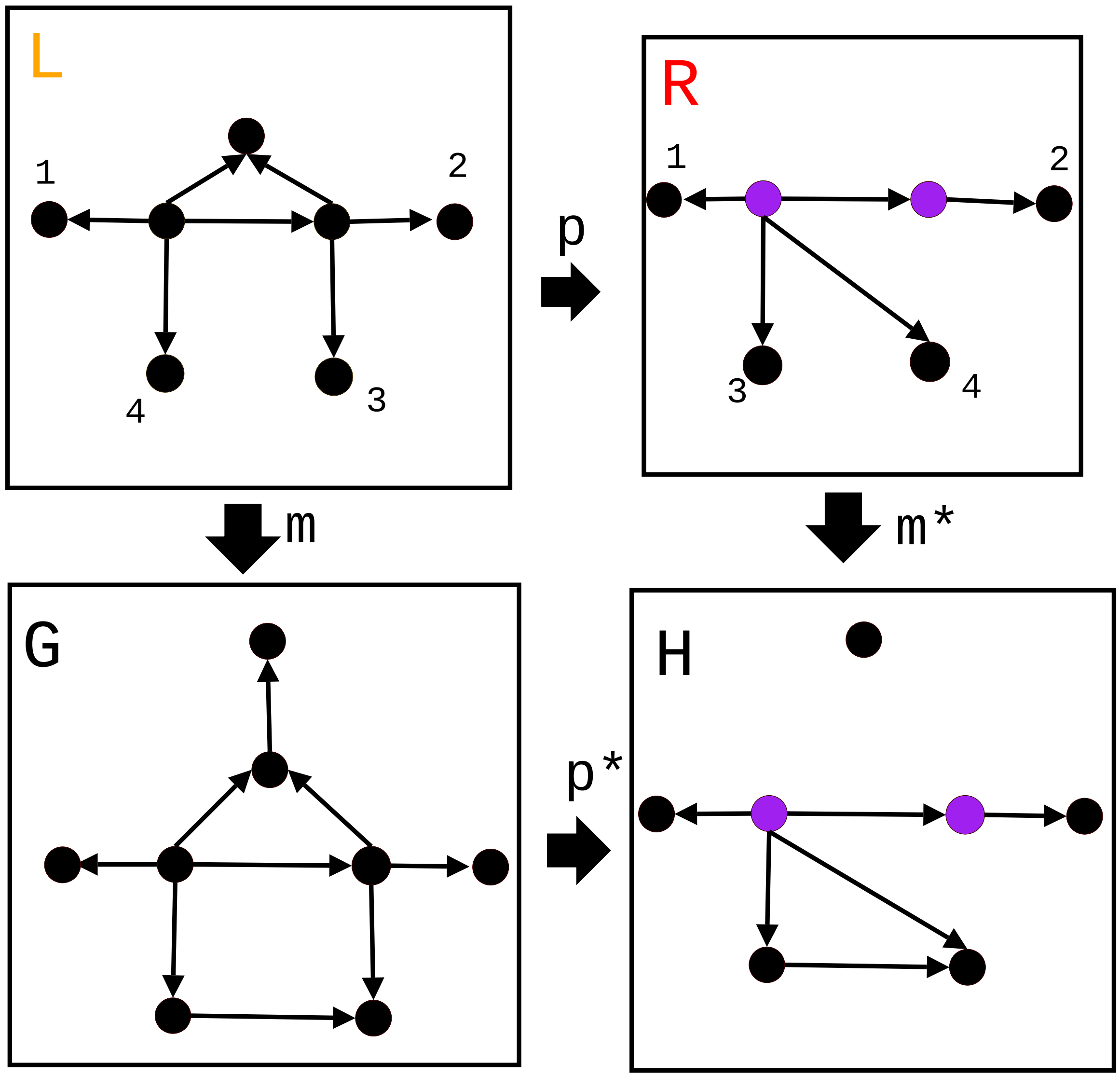


Figure 8: SPO graph transformation rule

References

[1] A. Corradini, H. Ehrig, R. Heckel, M. Korff, M. Löwe, L. Ribeiro, and A. Wagner. Algebraic approaches to graph transformation - part I: Basic concepts and double pushout approach. In G. Rozenberg, editor, *Handbook of Graph Grammars and Computing by Graph Transformation. Vol. I: Foundations*, chapter 4, pages 247–312. World Scientific, 1997.

[2] A. H. B. H. H.-J. K. S. K. D. P. A. S. G. T. M.Andries, Gregor Engels. *Graph Transformation for specification and programming*. PhD thesis, 1996.

[3] B. T. Messmer and H. Bunke. Subgraph isomorphism in polynomial time. Technical Report IAM 95-003, 1995.

[4] J. Ullman. An algorithm for subgraph isomorphism. *Journal of the ACM*, 23:31–42, 1976.