



# Solving Dynamic Sets of Non-Causal Equations

Andrew Casey



# Non-Causal Equations in $\mu$ Modelica



What is a non-causal  
equation?



# Examples

$$x = a + \sqrt{b^2 - 4ac}$$

$$F = ma$$

$$\frac{dx}{dt} = x$$

$$g = \frac{GM}{r^2}$$



# Solving

1. Simplify
2. Canonicalize
3. Assign Causality
4. Order Equations
5. Solve



# Example

- $a + b + c = 0$
- $a + c + d^2 = 2 * (a + 2 * c)$
- $(c^2) / c + 4 = d + 20$
- $d = -5$

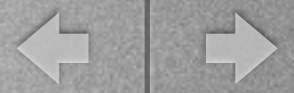


# Simplify

- Move everything to the LHS
- Re-express with fewer operators
- Apply simplification rules
- Fold constants
- Distribute constants
- Gather like terms

Xu - The Design and Implementation of the  $\mu$ Modelica compiler

Vangheluwe et al - An algorithm to implement a canonical representation of algebraic expressions and equations in AToM<sup>3</sup>



# Simplify

1.  $0 + expr \rightarrow expr$
2.  $0 * expr \rightarrow 0$
3.  $1 * expr \rightarrow expr$
4.  $expr^1 \rightarrow expr$
5.  $expr^0 \rightarrow 1$
6.  $1^{expr} \rightarrow 1$
7.  $a * expr + b * expr \rightarrow (a + b) * expr$
8.  $(expr_1^{expr_2})^{expr_3} \rightarrow expr_1^{expr_2 * expr_3}$



# Simplify

$$2 * (a + 2 * c) + d^2 = a + c$$



# Simplify

$$2 * (a + 2 * c) + d^2 - (a + c) = 0$$

Move RHS to LHS



# Simplify

$$2 * (a + 2 * c) + d^2 - (a + c)$$

Drop RHS



# Simplify

$+(* (2, +(a, *(2, c))), ^ (d, 2), * (-1, +(a, c)))$

Rewrite operators



# Simplify

$+(* (2, a), *(2, 2, c), ^ (d, 2), +(* (-1, a), * (-1, c)))$

Distribute constant multipliers



# Simplify

$+(* (2, a), *(4, c), ^ (d, 2), +(* (-1, a), * (-1, c)))$

Fold constants



# Simplify

$$+(*(+2, -1), a), *(+4, -1), c), ^{(d, 2)}$$

Gather like terms



# Simplify

**$+(a, *(3, c), ^{(d, 2)})$**

Fold constants



# Simplify

- $+(a, b, c)$
- $+(a, *(3, c), ^{(d, 2)})$
- $+(c, *(-1, d), -16)$
- $+(d, 5)$



# Issues

- There is no “simplest” form
  - e.g.  $(x^{100} - y^{100}) / (x - y)$
- Some simplifications will not be performed
  - e.g.  $+(*(-1, x), *(-1, y))$
- Two expressions may be equal and yet not have the same canonical form
  - e.g.  $\sin(x + \pi/2) = \cos(x)$



# Canonicalize

1. Constants - numerical order
2. Names - alphabetical order
3. Sums - dictionary order
4. Products - dictionary order
5. Powers - dictionary order



# Canonicalize

- $+(a, b, c)$  ( $\alpha$ )
- $+(a, *(3, c), ^{(d, 2)})$  ( $\beta$ )
- $+(-16, c, *(-1, d))$  ( $\gamma$ )
- $+(5, d)$  ( $\delta$ )



# Causality Assignment

Build a flow graph

## **Nodes**

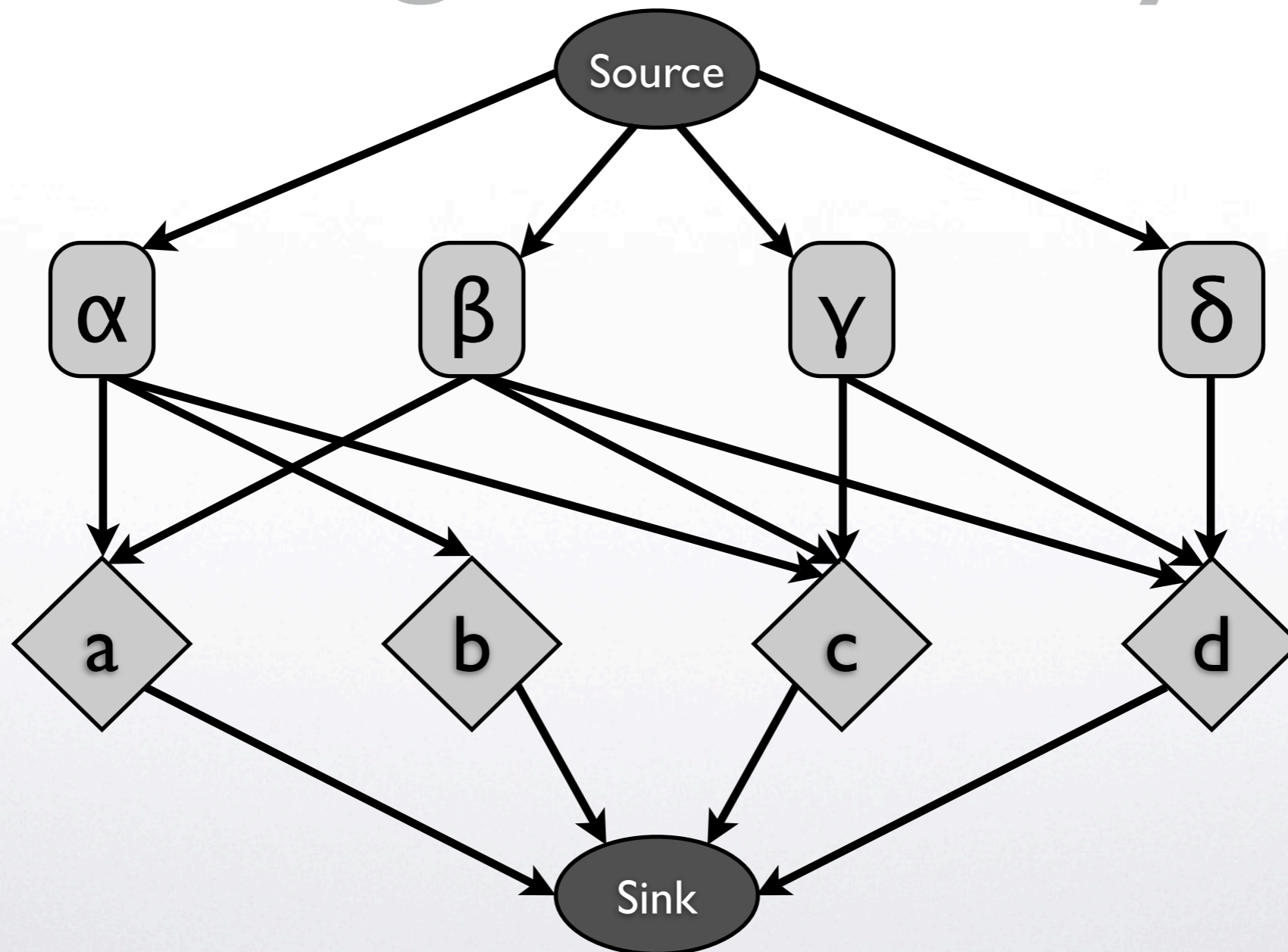
- Source
- Sink
- Equations
- Variables

## **Edges**

- Source to Equation
- Variable to Sink
- Equation to contained Variable



# Assign Causality





# Assign Causality

- Find a path from Source to Sink
- Reverse the edges
- Repeat

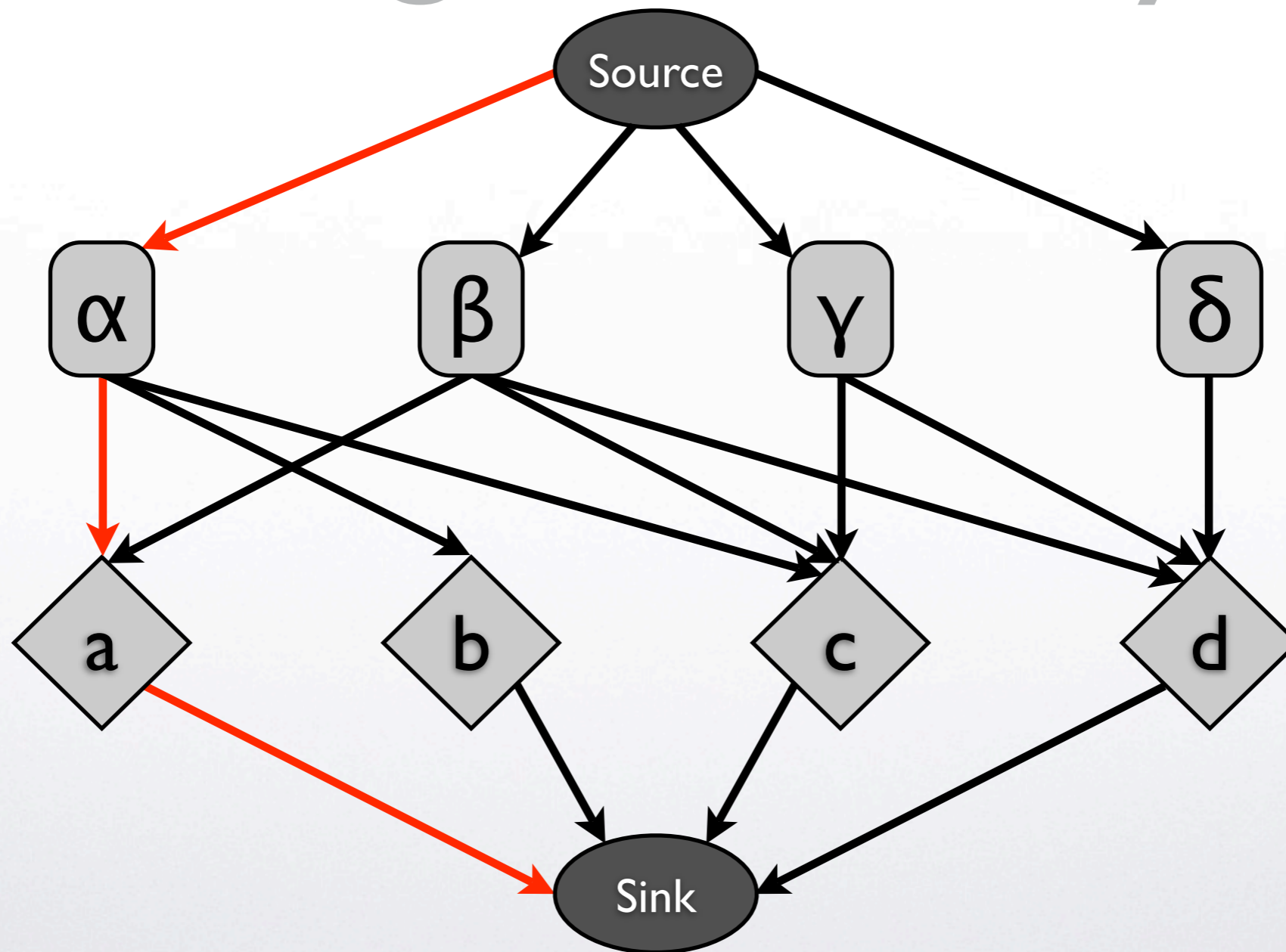
Xu - The Design and Implementation of the  $\mu$ Modelica compiler

Dinic - Algorithm for solution of a problem of maximum flow in a network with power estimation

Cormen et al - Introduction to Algorithms

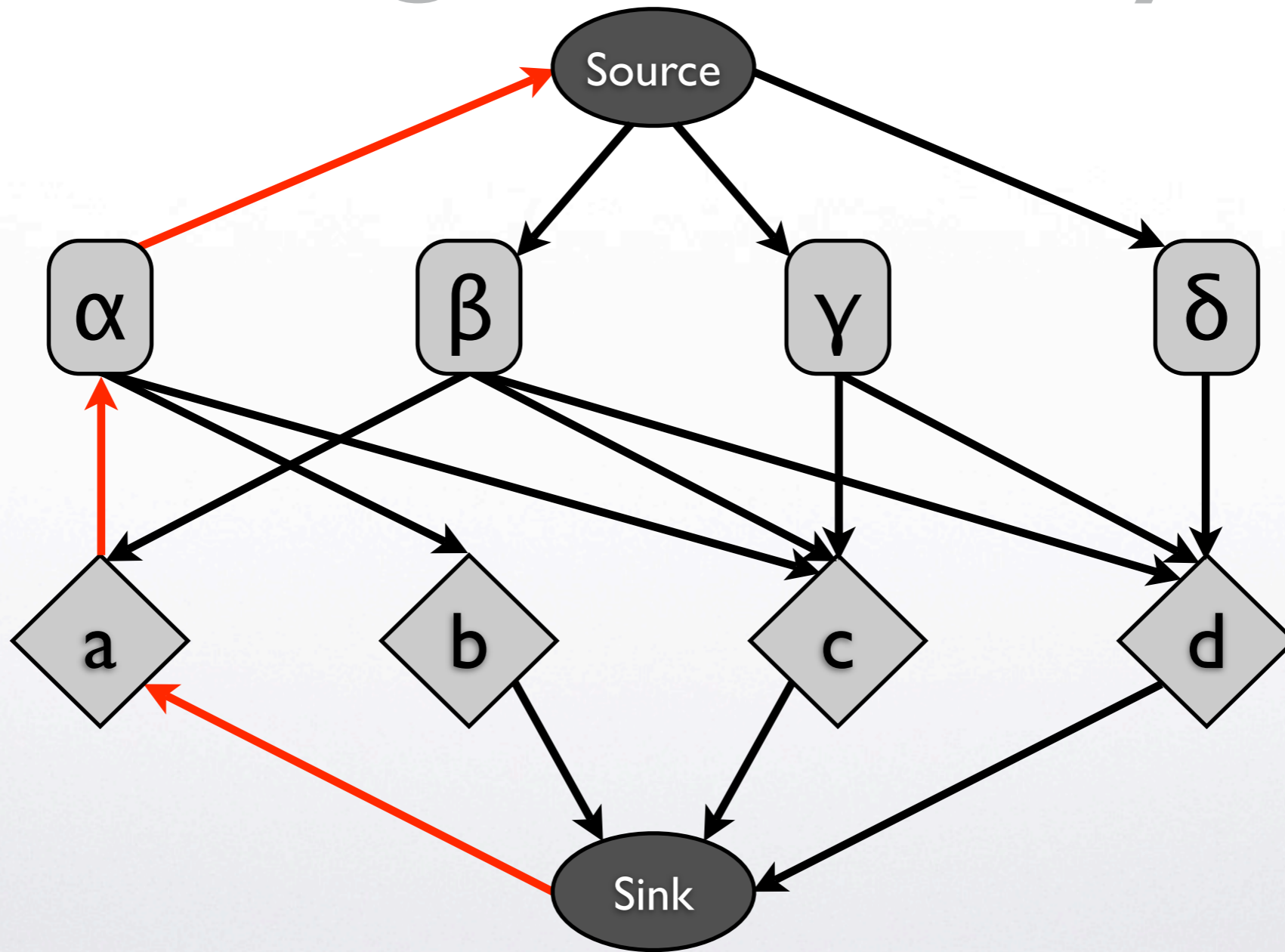


# Assign Causality



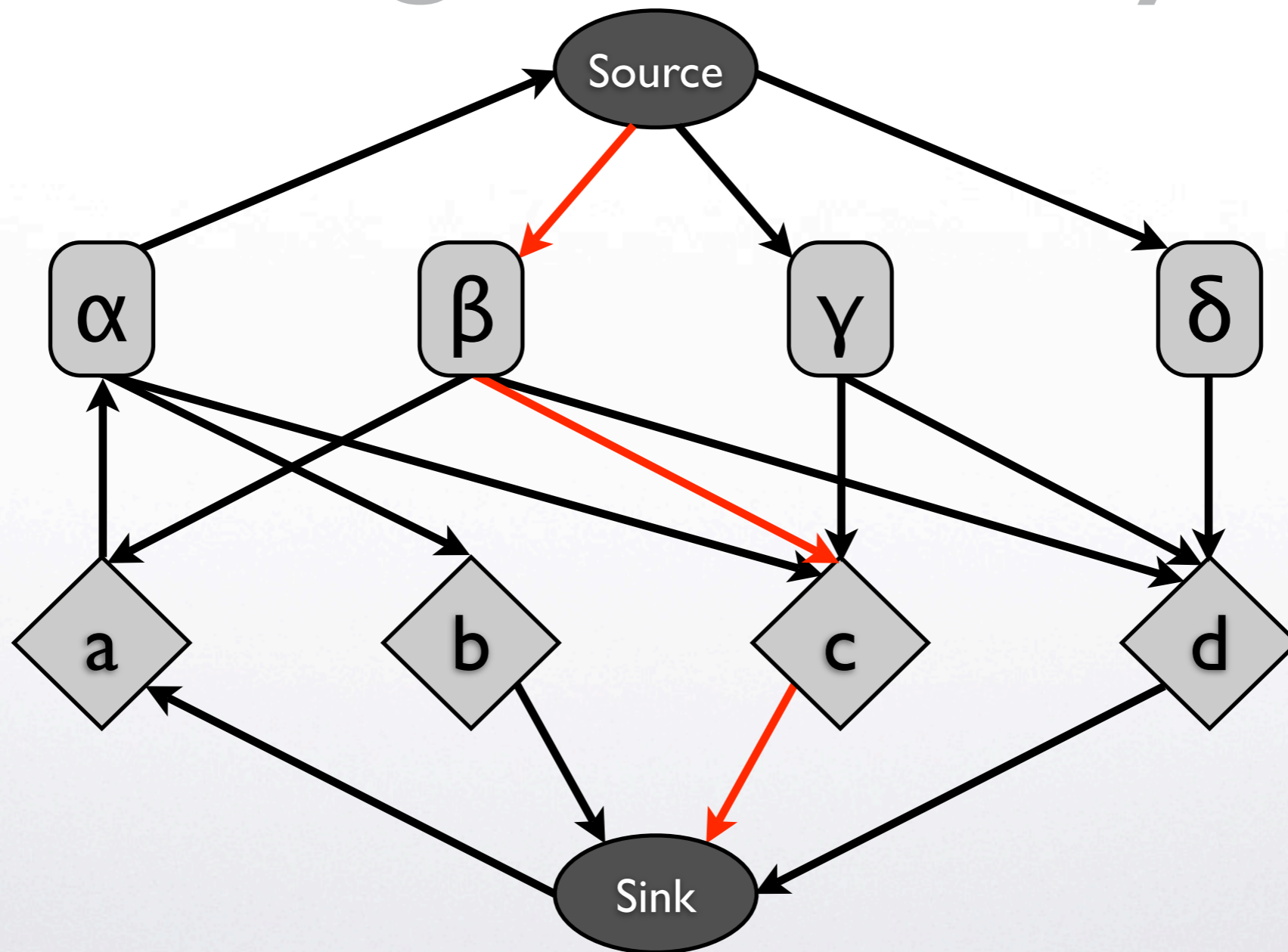


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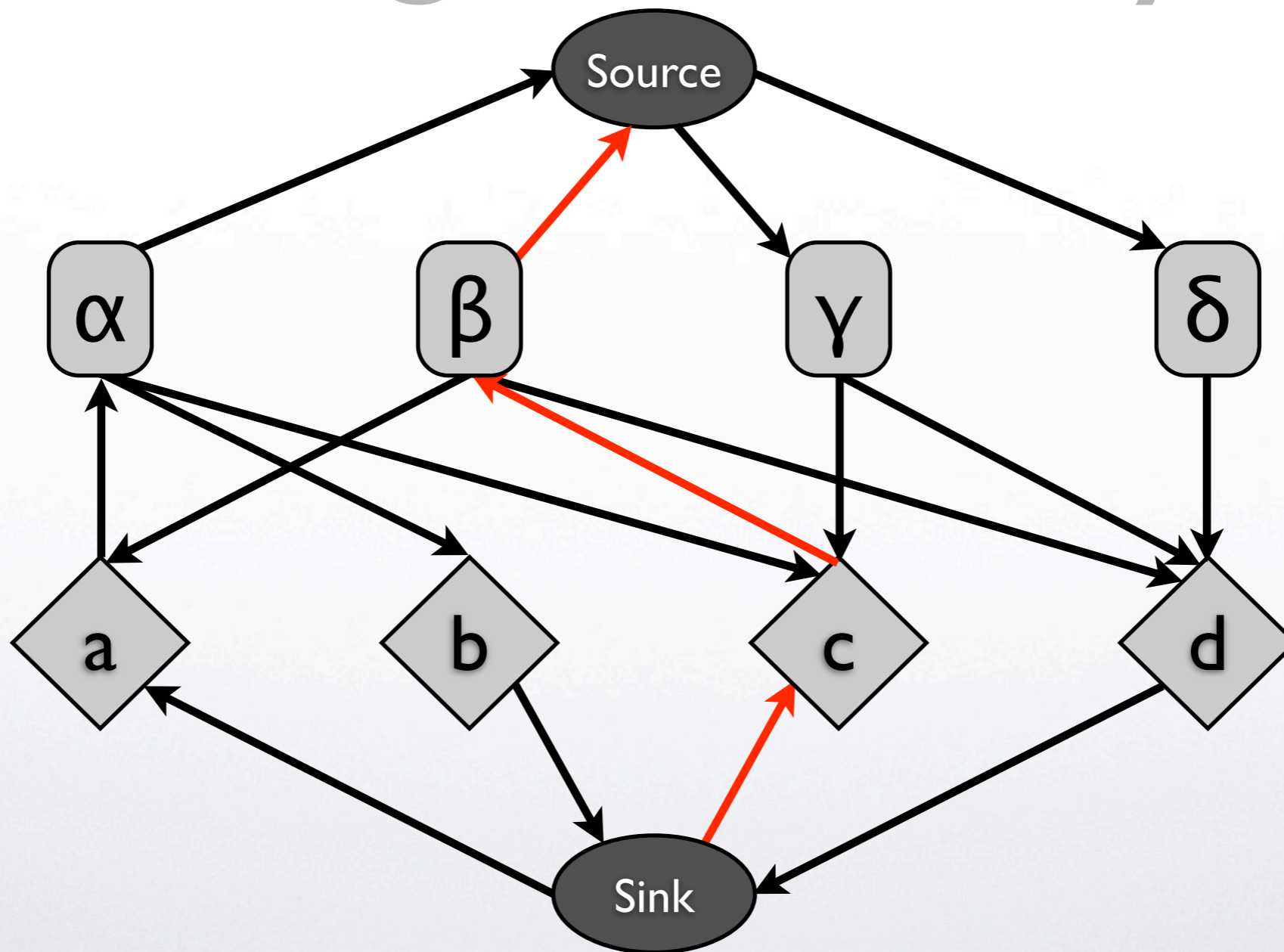


# Assign Causality



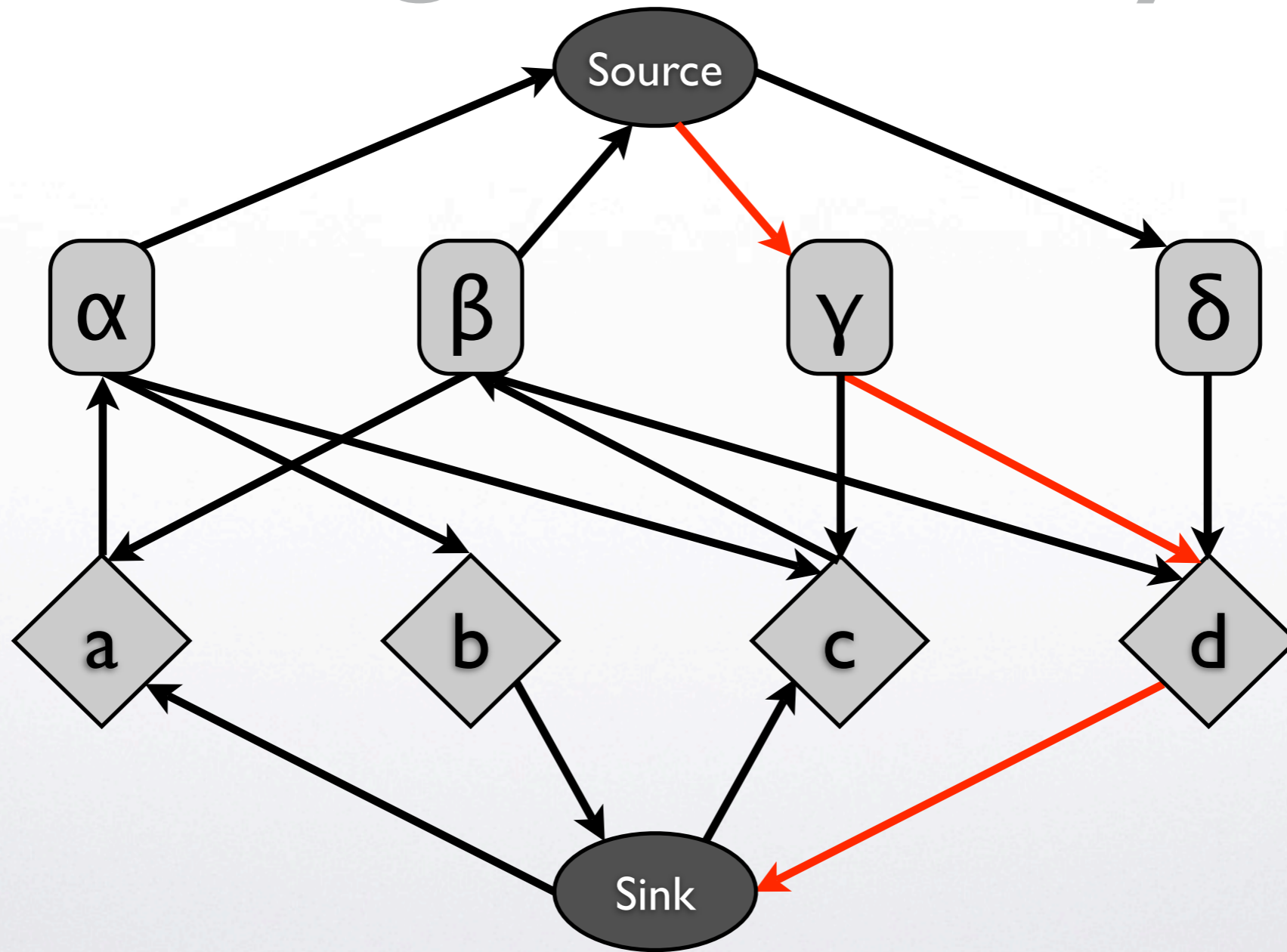


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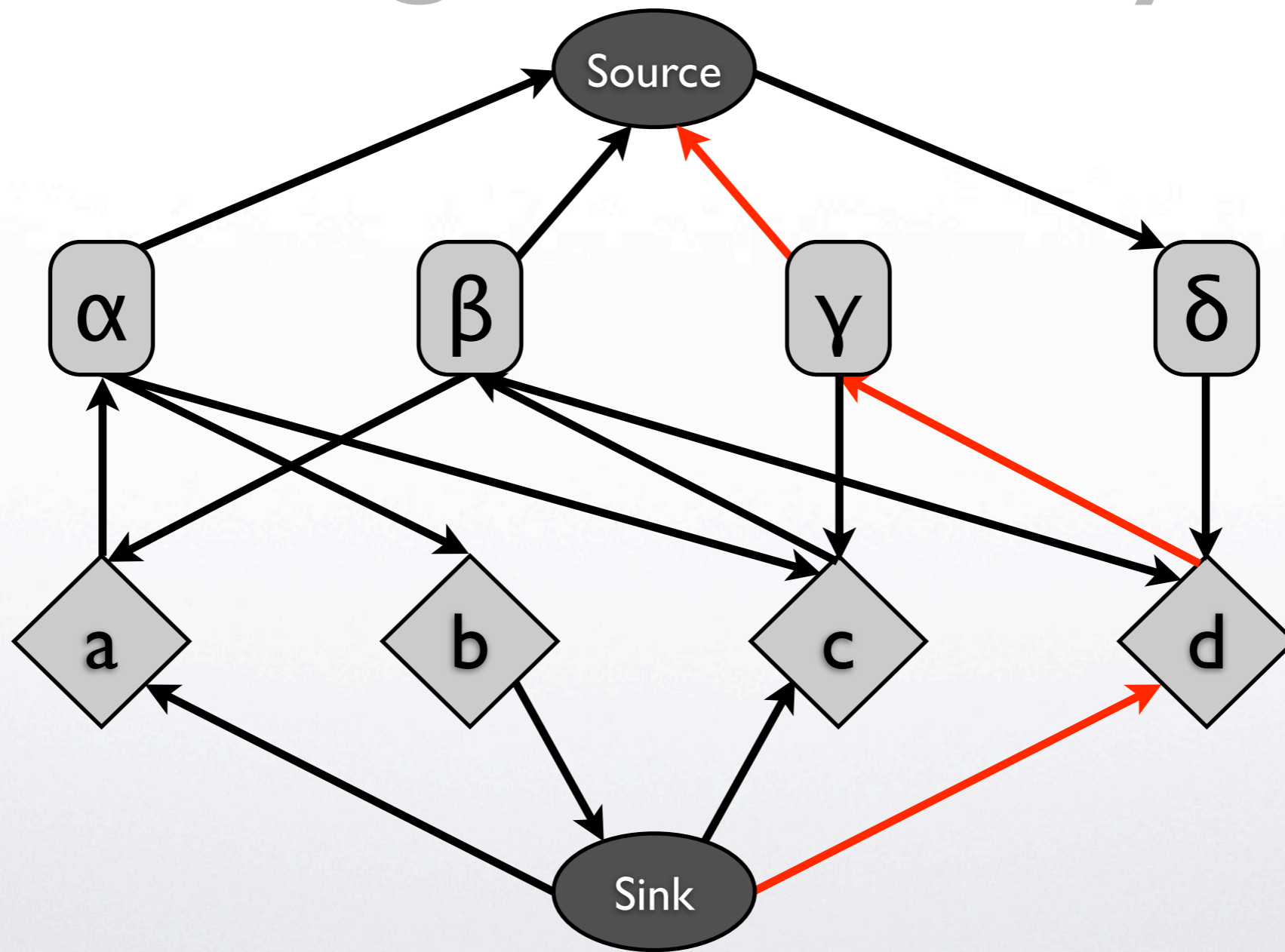


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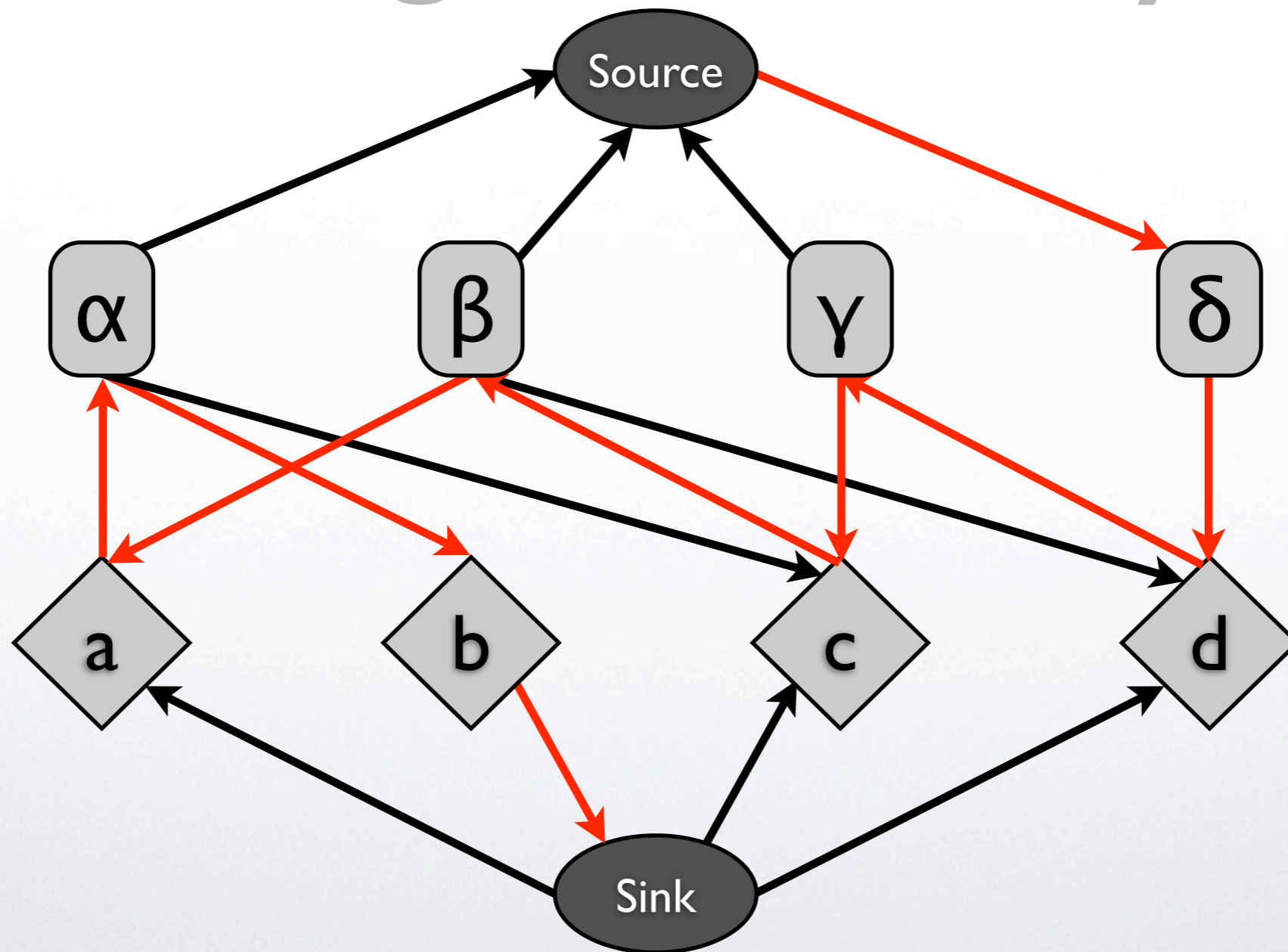


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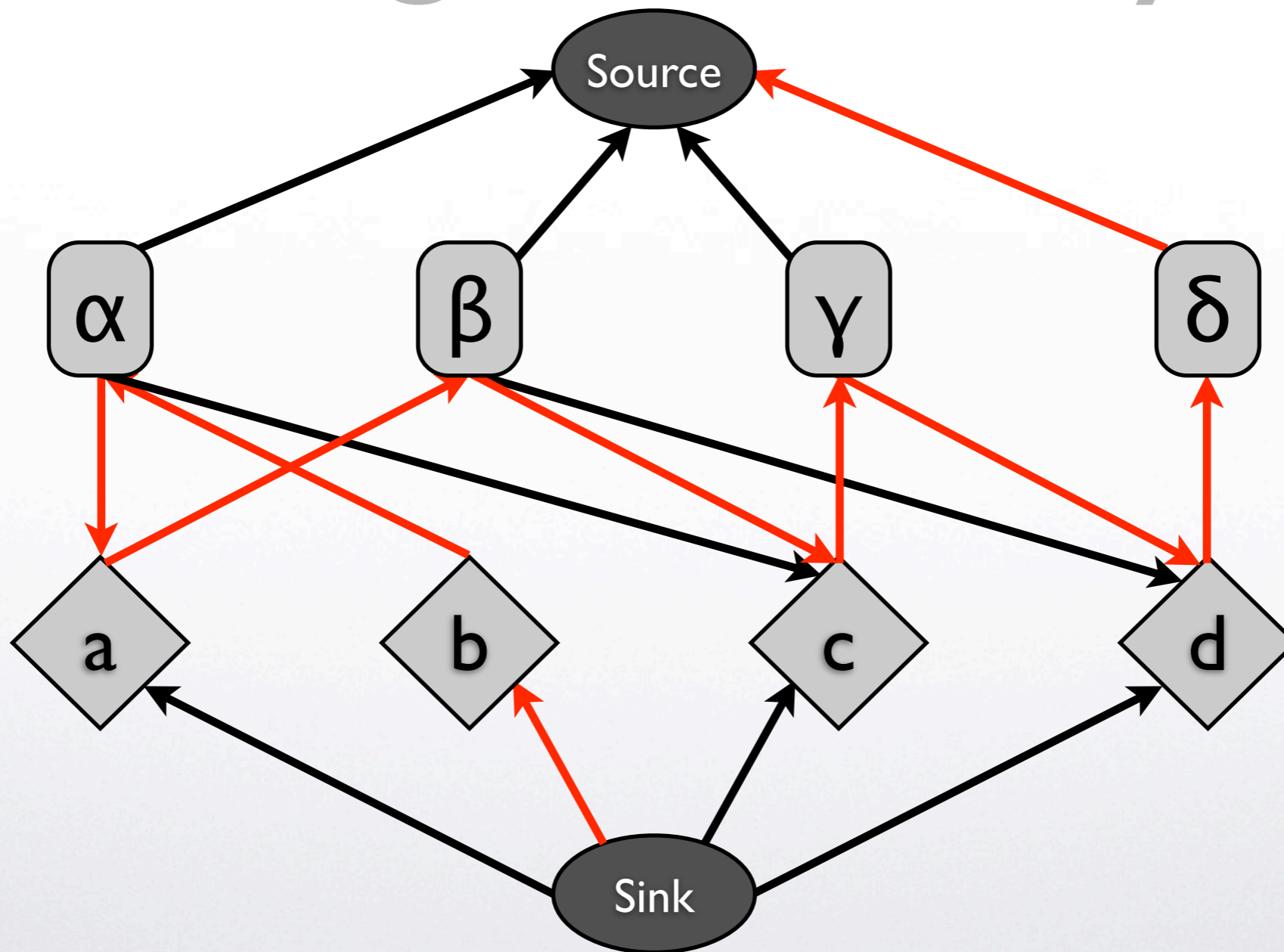


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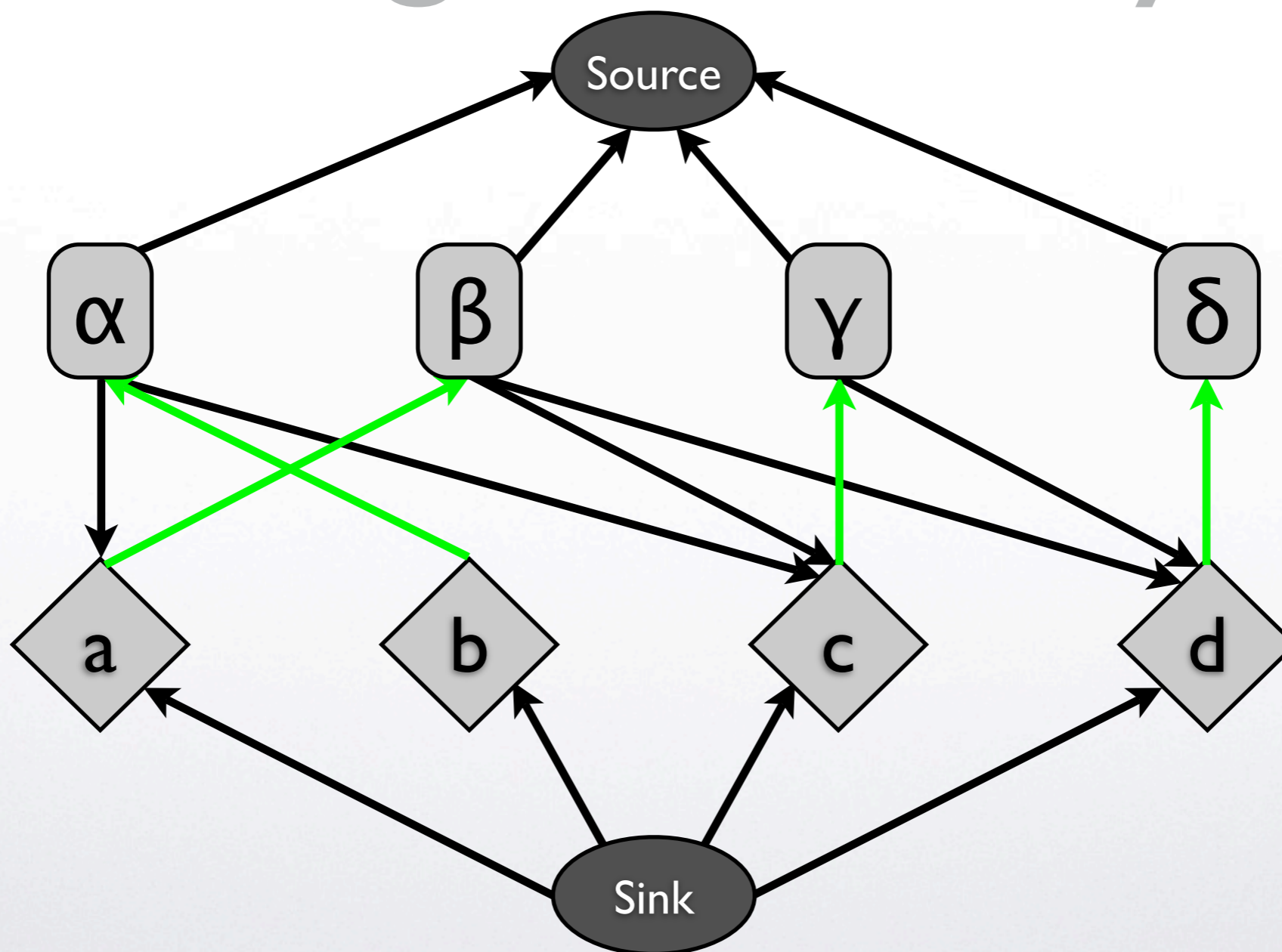


# Assign Causality





# Assign Causality





# Assign Causality

- $+(a, b, c)$  ( $\alpha$ )
- $+(a, *(3, c), ^{(d, 2)})$  ( $\beta$ )
- $+(-16, c, *(-1, d))$  ( $\gamma$ )
- $+(5, d)$  ( $\delta$ )



# Assign Causality

- $b = +(*(-1, a), *(-1, c))$  ( $\alpha$ )
- $a = +(*(-3, c), *(-1, \wedge(d, 2)))$  ( $\beta$ )
- $c = +(16, d)$  ( $\gamma$ )
- $d = -5$  ( $\delta$ )



# Issues

- Some equations do not have algebraic solutions
  - e.g.  $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$
- Solving can be non-trivial
  - e.g.  $(x + 1)^2 + (x - 1)^2 = 10$
- Systems may be over-/under-determined

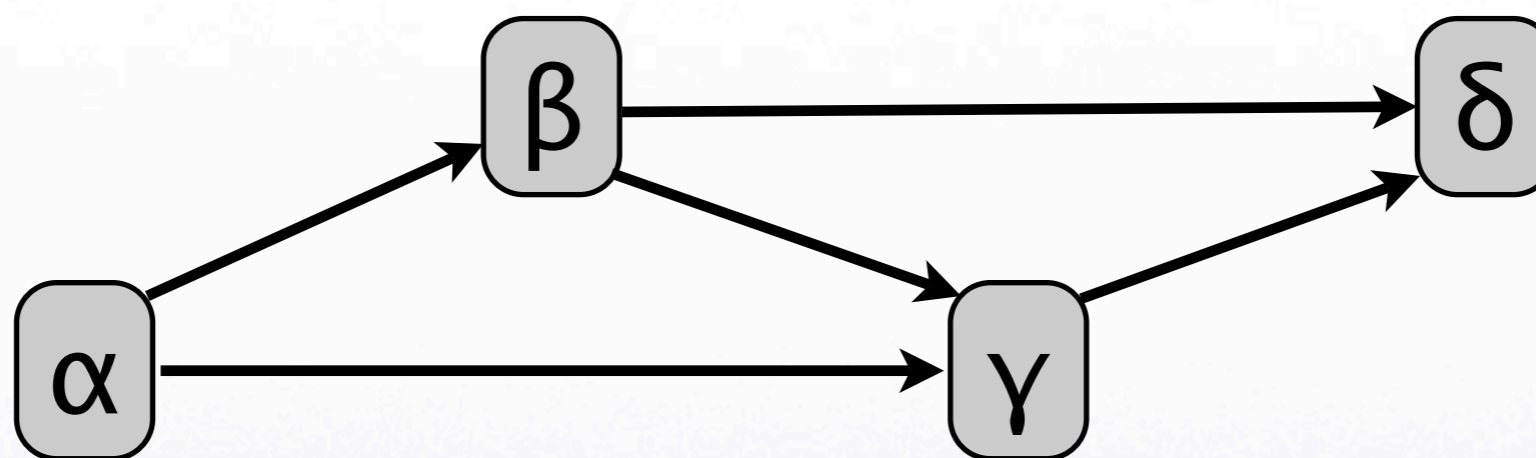


# Order Equations

- Build a dependency graph
- Topologically sort the strongly-connected components (SCCs)



# Order Equations





# Order Equations

1.  $d = -5$  ( $\delta$ )

2.  $c = +(16, d)$  ( $\gamma$ )

3.  $a = +(*(-3, c), *(-1, ^{(d, 2)}))$  ( $\beta$ )

4.  $b = +(*(-1, a), *(-1, c))$  ( $\alpha$ )



# Solve

1.  $d = -5$

2.  $c = +(16, d) = 11$

3.  $a = +(*(-3, c), *(-1, ^{(d, 2)})) = -58$

4.  $b = +(*(-1, a), *(-1, c)) = 47$



# Issues

- Equations may have multiple solutions
- SCCs may be under-determined
- How do dependencies work for equations without solvable variables?

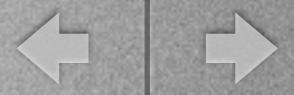


What if the user wants  
to add and remove  
equations at runtime?



# Dynamic Sets

- Cut some corners to make the system more responsive
- Try to update causality assignment and equation order dynamically
- Try to solve equations in parallel
- Try iterating a constant number of times instead of until fixed point



# References

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