Petri Nets

1. Finite State Automata
2. Petri net notation and definition (no dynamics)
3. Introducing State: Petri net marking
4. Petri net dynamics
5. Capacity Constrained Petri nets
6. Petri net models for . . .
   - FSA
   - Nondeterminism
   - Data Flow Computation
   - Communication Protocols

7. Queueing Systems
8. Petri nets vs. State Automata
9. Analysis of Petri nets
    - Boundedness
    - Liveness and Deadlock
    - State Reachability
    - State Coverability
    - Persistence
    - Language Recognition
10. The Coverability Tree
11. Extensions: colour, time, ...

Finite State Automaton

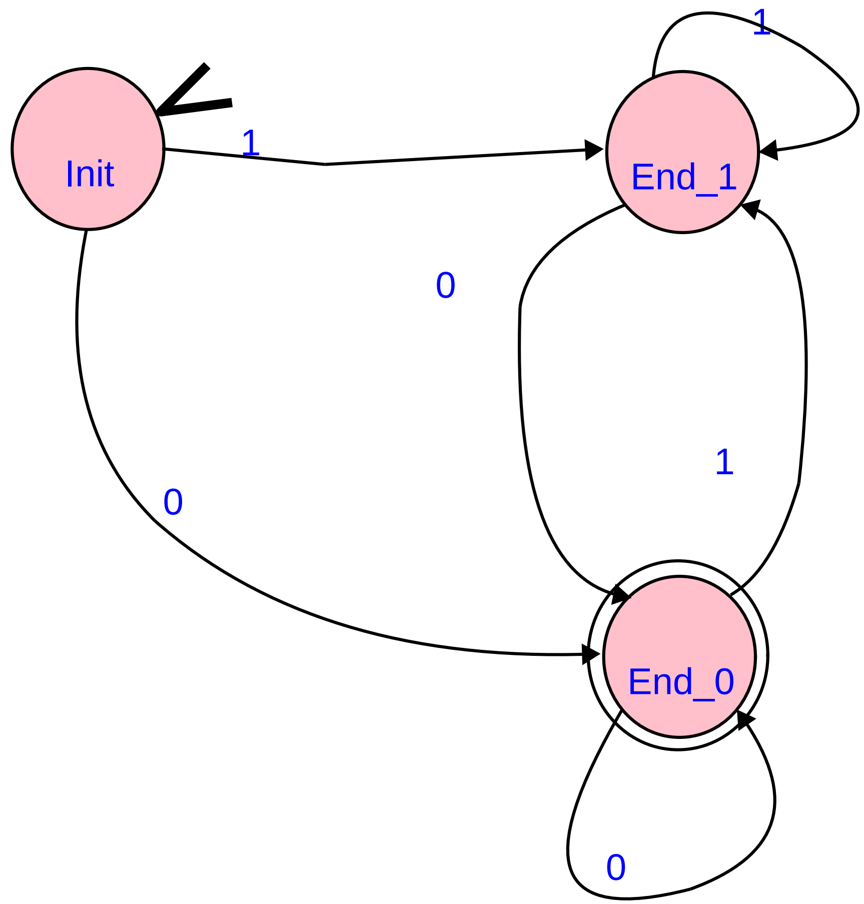
$$(E, X, f, x_0, F)$$

- E is a finite alphabet
- X is a finite state set
- f is a state transition function, $f : X \times E \to X$
- x_0 is an initial state, $x_0 \in X$
- F is the set of final states

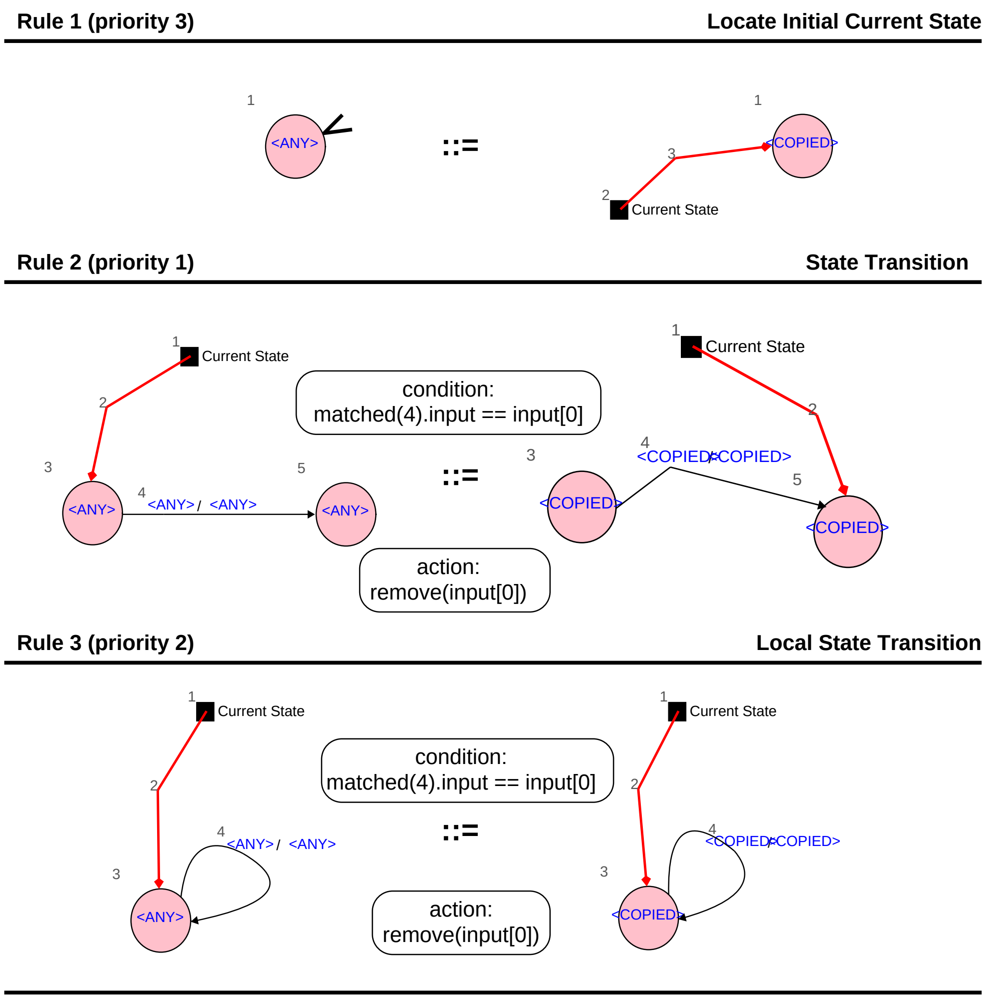
Dynamics (x' is next state):

$$x' = f(x, e)$$

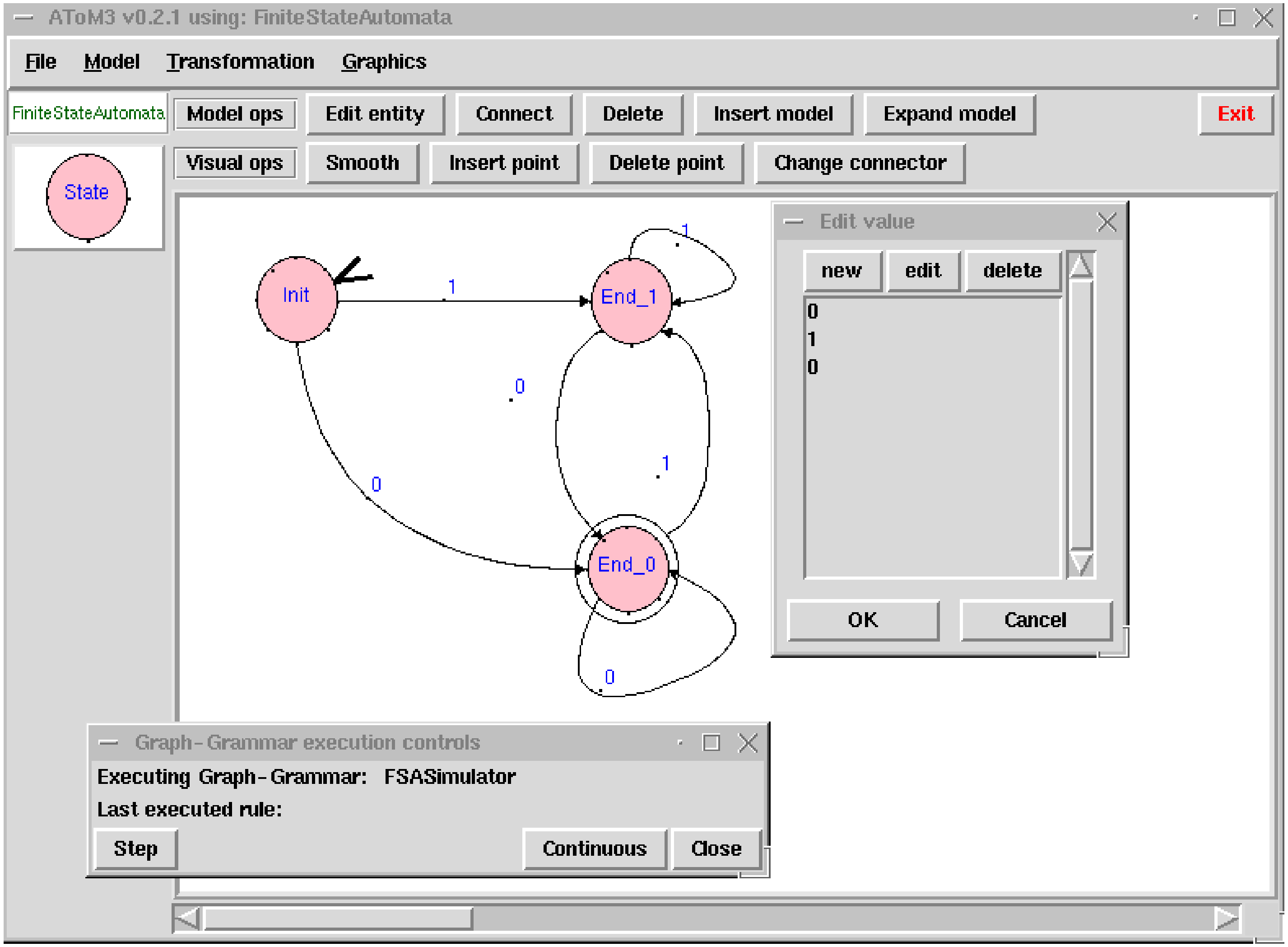
FSA graphical/visual notation: State Transition Diagram

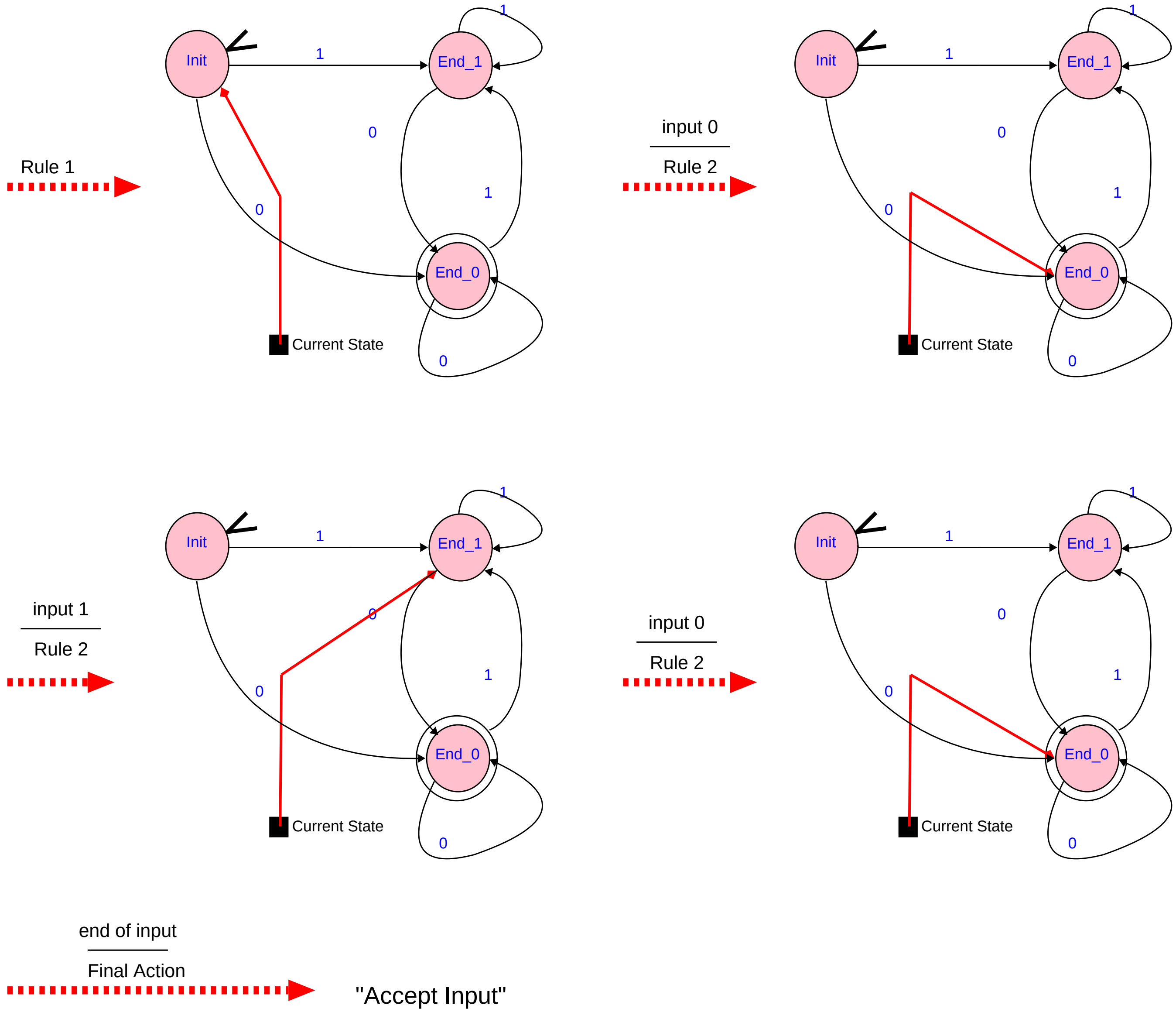


FSA Operational Semantics



Simulation steps




Rule 1
Init
End_1
End_0
Current State
input 0
Rule 2
input 1
Rule 2
input 0
Rule 2
end of input
Final Action
"Accept Input"

State Automaton

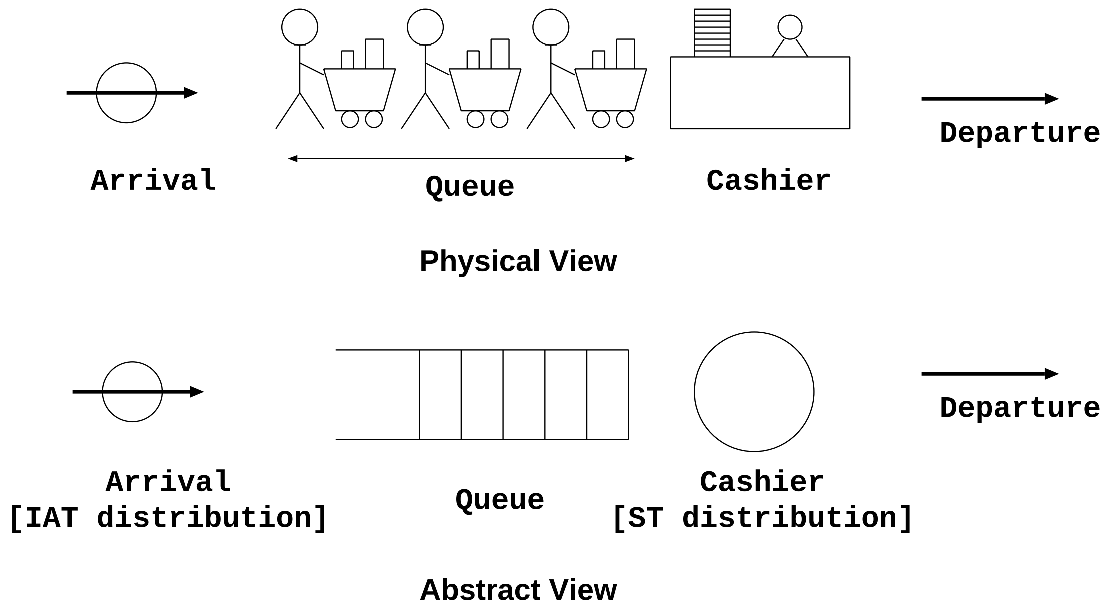
$$(E, X, \Gamma, f, x_0)$$

- E is a countable event set
- X is a countable state space
- $\Gamma(x)$ is the set of feasible or enabled events
  $x \in X, \Gamma(x) \subseteq E$
- f is a state transition function,
  $f : X \times E \rightarrow X$, only defined for $e \in \Gamma(x)$
- x_0 is an initial state, $x_0 \in X$

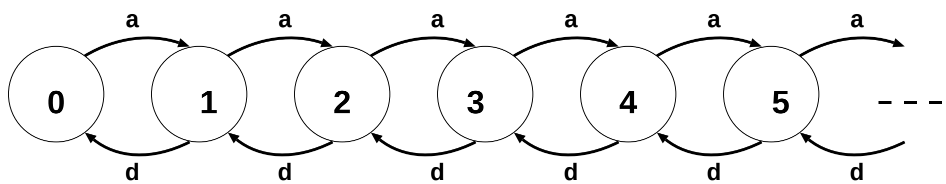
$$(E, X, \Gamma, f)$$

omits x_0 and describes a class of State Automata.

State Automata for Queueing Systems



State Automata for Queueing Systems: customer centered



$$E = \{a, d\}$$

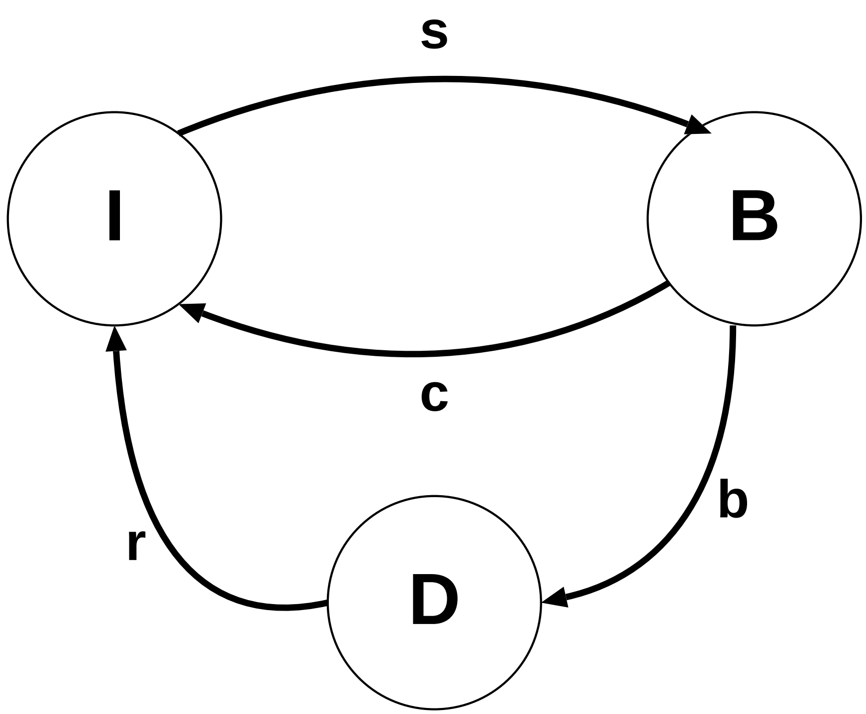
$$X = \{0, 1, 2, \ldots\}$$

$$\Gamma(x) = \{a, d\}, \forall x > 0; \Gamma(0) = \{a\}$$

$$f(x, a) = x + 1, \forall x \geq 0$$

$$f(x, d) = x - 1, \forall x > 0$$

State Automata for Queueing Systems: server centered (with breakdown)



State Automata for Queueing Systems: server centered (with breakdown)

$$E = \{s, c, b, r\}$$

Events: s denotes service starts, c denotes service completes, b denotes breakdown, r denotes repair.

$$X = \{I, B, D\}$$

State: I denotes idle, B denotes busy, D denotes broken down.

$$\Gamma(I) = \{s\}, \Gamma(B) = \{c, b\}, \Gamma(D) = \{r\}$$

$$f(I, s) = B, f(B, c) = I, f(B, b) = D, f(D, r) = I$$

Limitiations/extensions of State Automata

- Adding time ?
- Hierarchical modelling ?
- Concurrency by means of $\times$
- States are represented explicitly
- Specifying control logic, synchronisation ?

Petri nets

- Formalism similar to FSA
- Graphical/Visual notation
- C.A. Petri 1960s
- Additions to FSA:
  - Explicitly (graphically/visually) represent when event is enabled
    $\rightarrow$ describe control logic
  - Elegant notation of concurrency
  - Express non-determinism

Petri net notation and definition (no dynamics)

$$(P, T, A, w)$$

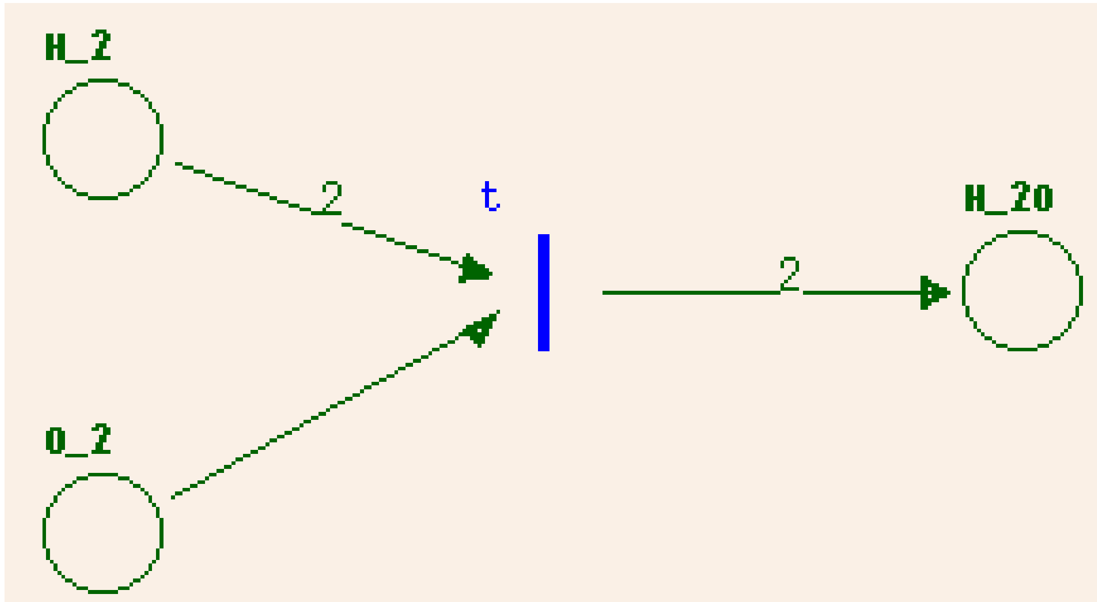
- $P = \{p_1, p_2, \ldots\}$ is a finite set of *places*
- $T = \{t_1, t_2, \ldots\}$ is a finite set of *transitions*
- $A \subseteq (P \times T) \cup (T \times P)$ is a set of *arcs*
- $w : A \to \mathbb{N}$ is a *weight function*

Note: no need for *countable* P and T.

Derived Entities

- $I(t_j) = \{p_i : (p_i, t_j) \in A\}$ set of *input places* to transition t_j ($\equiv$ conditions for transition)
- $O(t_j) = \{p_i : (t_j, p_i) \in A\}$ set of *output places* from transition t_j ($\equiv$ affected by transition)
- Transitions $\equiv$ events
- similarly: input- and output-transitions for p_i
- graphical/visual representation: *Petri net graph* (multigraph)

Example Petri net



- $P = \{H_2, O_2, H_2O\}$
- $T = \{t\}$
- $A = \{(H_2, t), (O_2, t), (t, H_2O)\}$
- $w((H_2, t)) = 2, w((O_2, t)) = 1, , w((t, H_2O)) = 2$

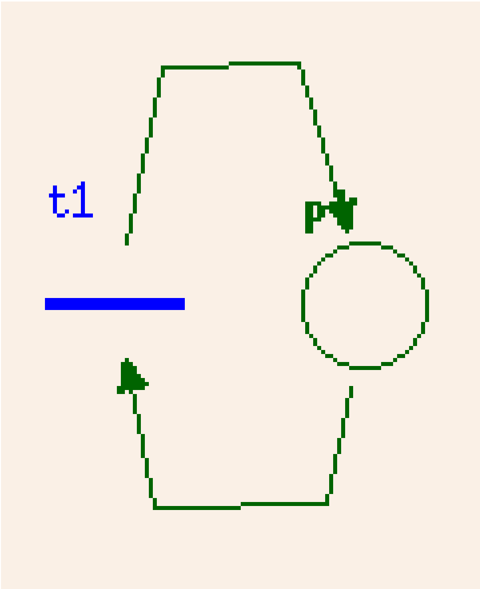
Pure Petri net

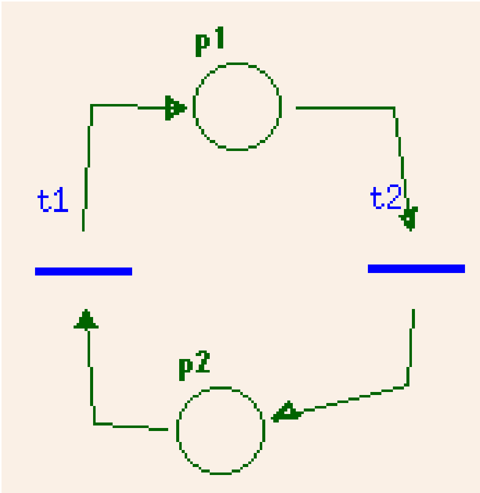
- No self-loops:

$$\nexists p_i \in P, t_j \in T : (p_i, t_j) \in A, (t_j, p_i) \in A$$

- Can convert impure to pure Petri net

Impure to Pure Petri net





Introducing State: Petri net Markings

- Conditions met ? Use *tokens* in places
- Token assignment $\equiv$ *marking* x

$$x : P \rightarrow \mathbb{N}$$

- A marked Petri net

$$(P, T, A, w, x_0)$$

  x_0 is the *initial marking*

- The *state* **x** of a marked Petri net

$$\mathbf{x} = [x(p_1), x(p_2), \ldots, x(p_n)]$$

  Number of tokens need not be bounded (cfr. State Automata states).

State Space of Marked Petri net

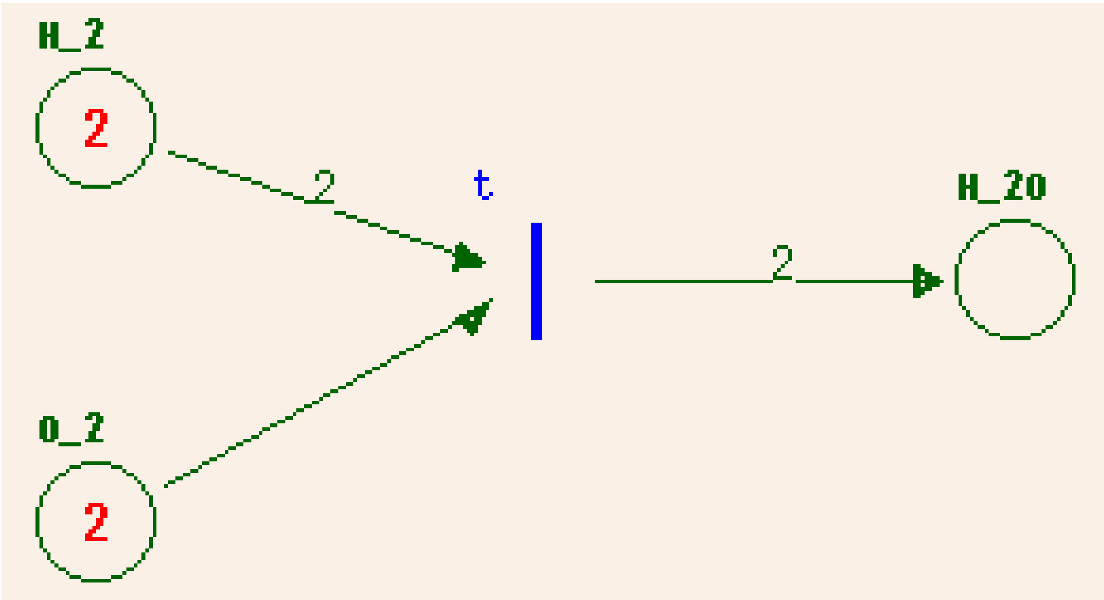
- All n-dimensional vectors of nonnegative integer markings

$$X = \mathbb{N}^n$$

- Transition $t_j \in T$ is *enabled* if

$$x(p_i) \geq w(p_i, t_j), \forall p_i \in I(t_j)$$

Example with marking, enabled



Petri Net Dynamics

State Transition Function f of marked Petri net (P, T, A, w, x_0)

$$f : \mathbb{N}^n \times T \rightarrow \mathbb{N}^n$$

is defined for transition $t_j \in T$ if and only if

$$x(p_i) \geq w(p_i, t_j), \forall p_i \in I(t_j)$$

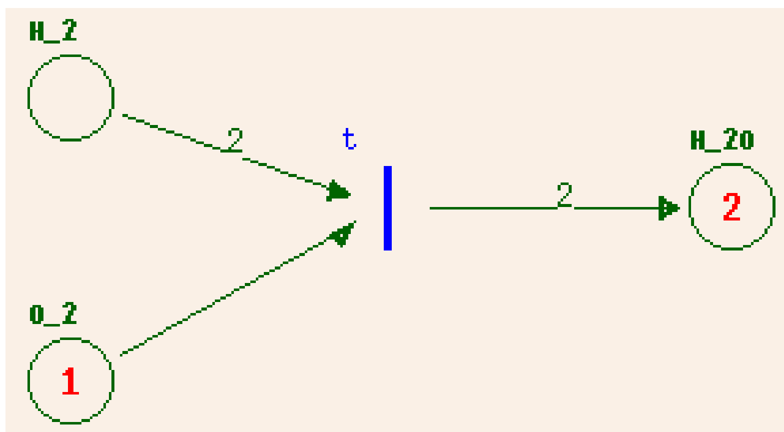
If $f(\mathbf{x}, t_j)$ is defined, set $\mathbf{x}' = f(\mathbf{x}, t_j)$ where

$$x'(p_i) = x(p_i) - w(p_i, t_j) + w(t_j, p_i)$$

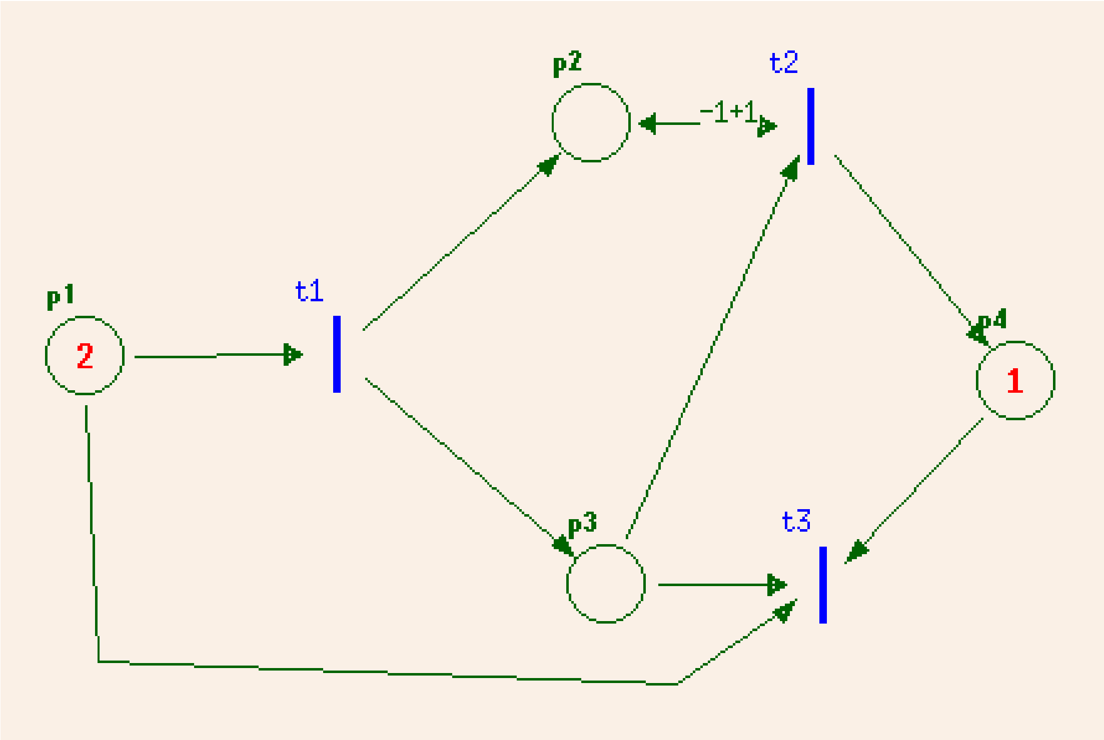
- State transition function f based on *structure* of Petri net
- Number of tokens *need not be conserved* (but can)

Example “firing”

- Use PNS tool `http://www.ee.uwa.edu.au/ braunl/pns/`
- Select Sequential Manual execution
- Transition: $[2, 2, 0] \rightarrow [0, 1, 2]$

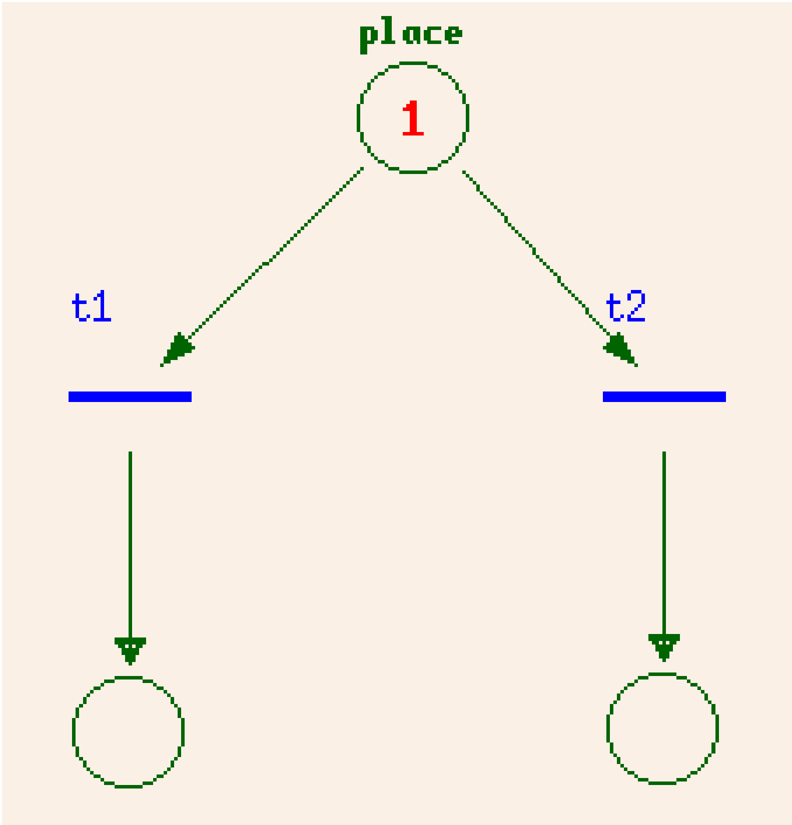


Example



- order of firing not determined (due to untimed model)
- selfloop
- “dead” net

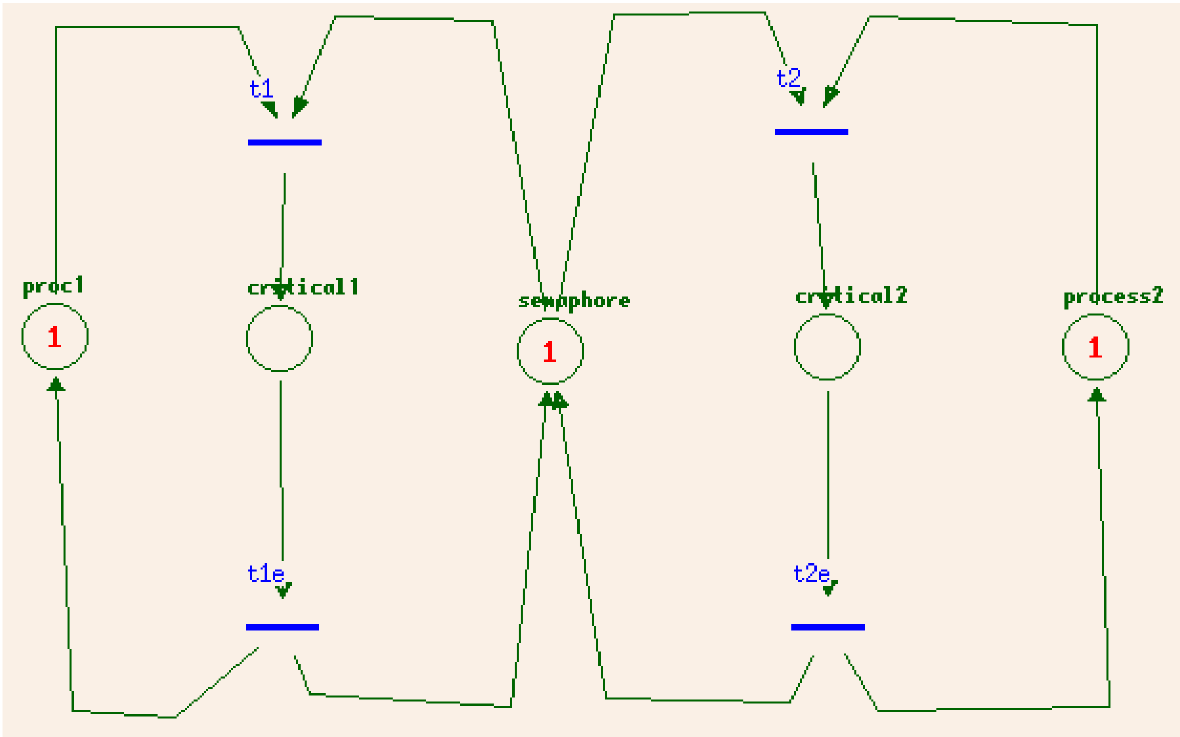
Conflict, choice, decision



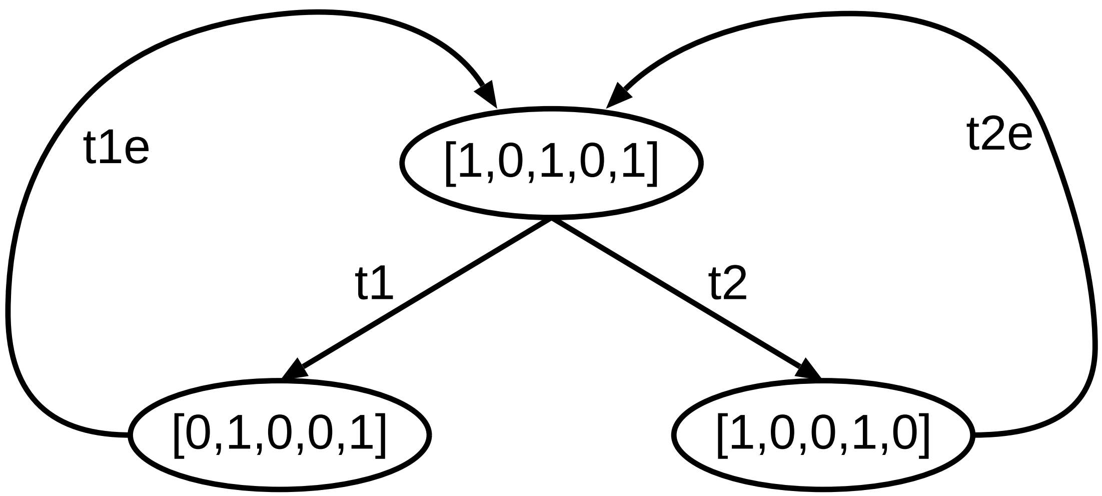
Semantics

- *sequential* vs. *parallel*
- Handle nondeterminism:
  1. User choice
  2. Priorities
  3. Probabilities (Monte Carlo)
  4. Reachability Graph (enumerate all choices)

Application: Critical Section



Reachability Graph



Algebraic Description of Dynamics

- Firing vector $\mathbf{u}$: transition j firing

$$\mathbf{u} = [0, 0, \ldots, 1, 0, \ldots, 0]$$

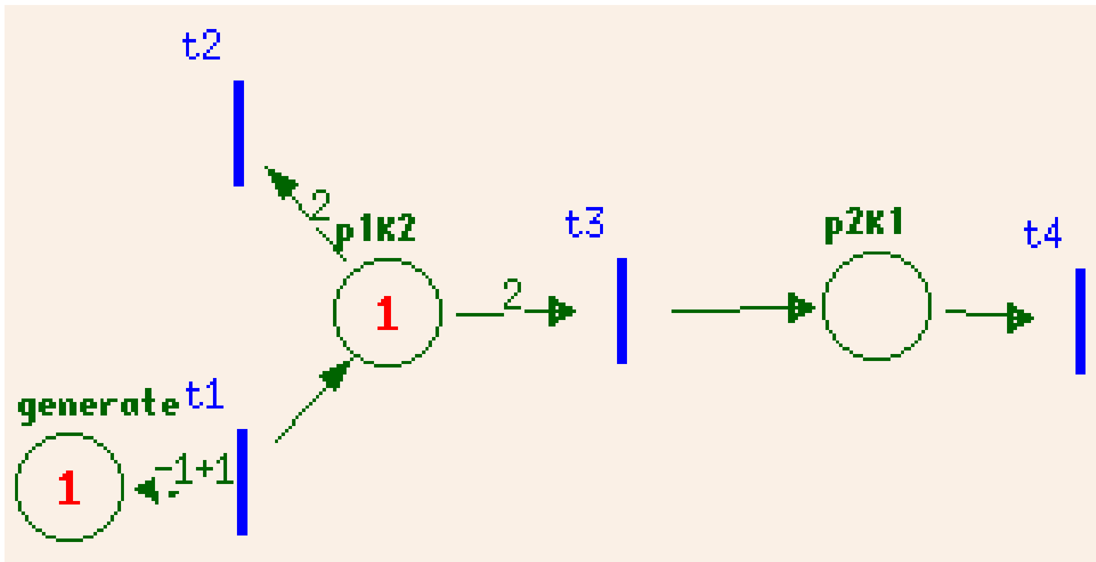
- Incidence matrix $\mathbf{A}$:

$$a_{ji} = w(t_j, p_i) - w(p_i, t_j)$$

- State Equation

$$\mathbf{x}' = \mathbf{x} + \mathbf{u}\mathbf{A}$$

Infinite Capacity Petri net

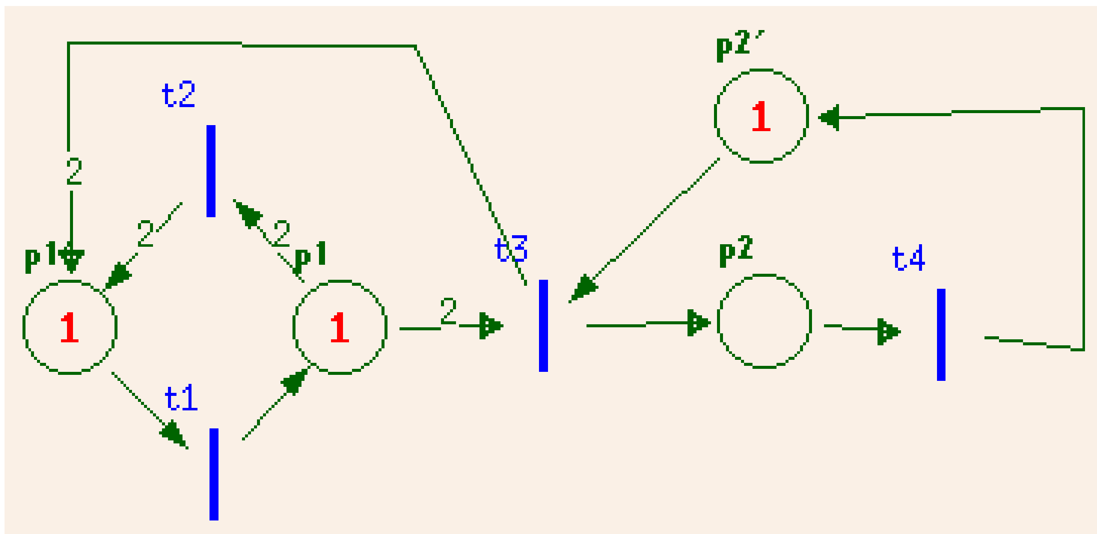


- Add Capacity Constraint: $K : P \rightarrow \mathbb{N}$
- New transition rule

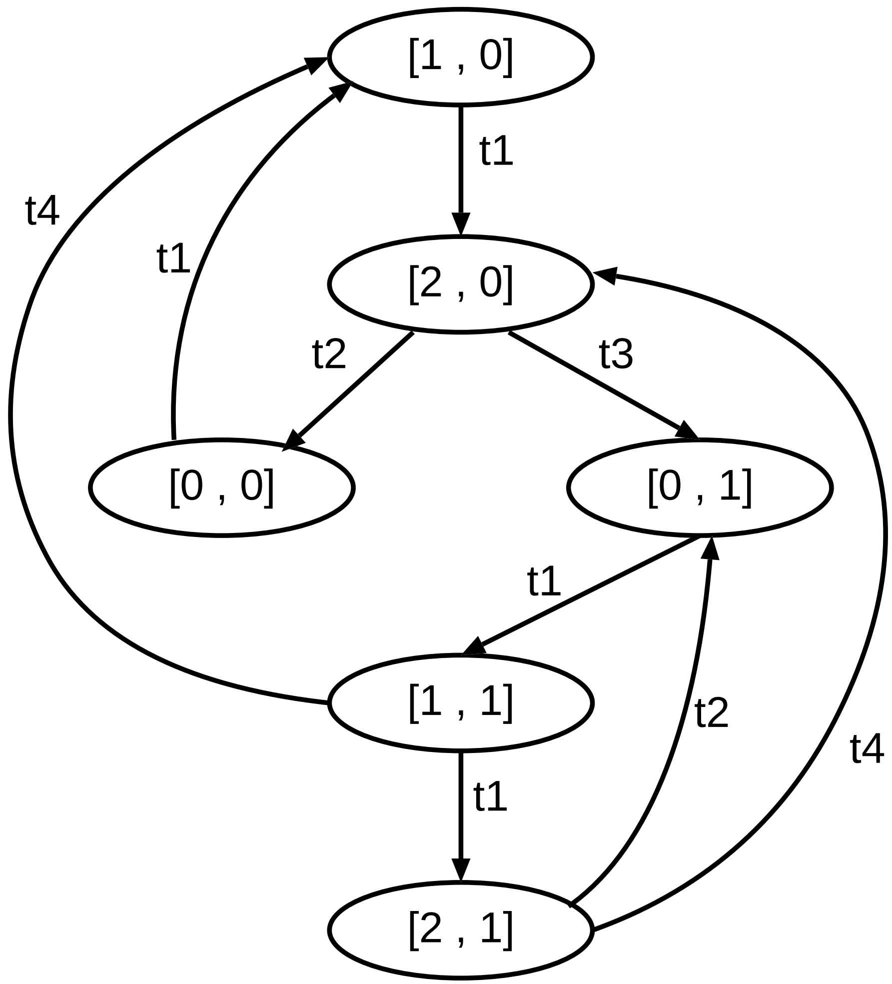
Can transform to infinite capacity net

1. Add complimentary place p' with initial marking $x_0(p') = K(p)$
2. Between each transition t and complimentary places p'
   - add arcs (t, p') or (p', t) where
   - $w(t, p') = w(p, t)$
   - $w(p', t) = w(t, p)$

Capacity Constrained Petri net

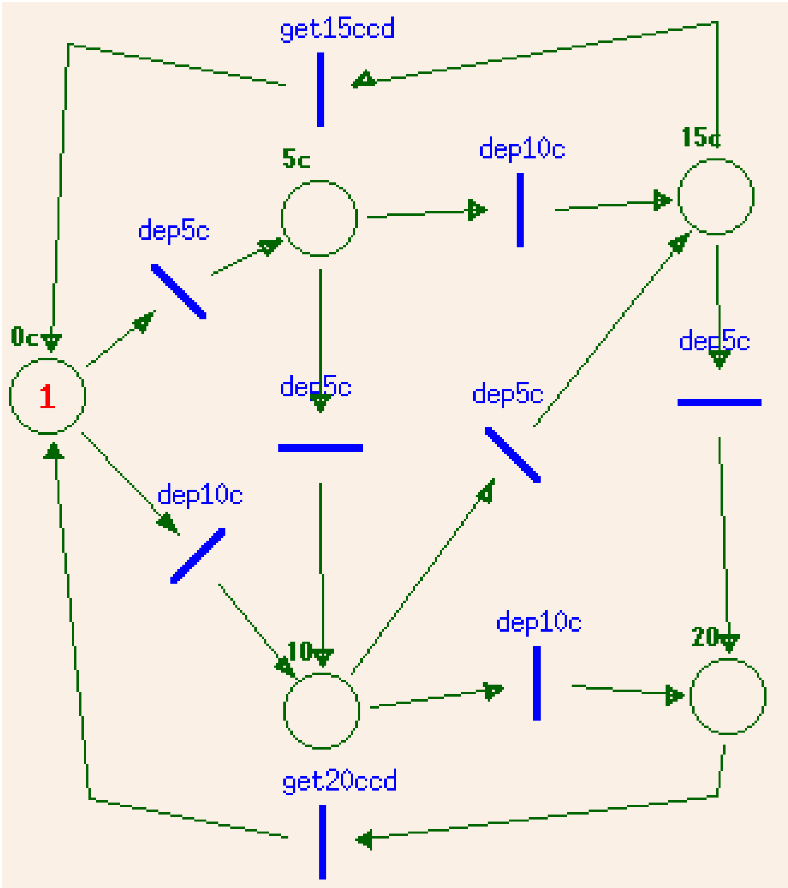


Equivalence proof: use Reachability Graph



[p1K2 , p2K1]

Petri net as State Machine



Representing a Petri net as a State Machine

Construct Reachability Graph

- Reachability Graph is State Machine
- States are tuples $(p_1, p_2, \ldots, p_n)$
- Events correspond to t_i firing
- May be infinite

Representing a State Machine as a Petri net

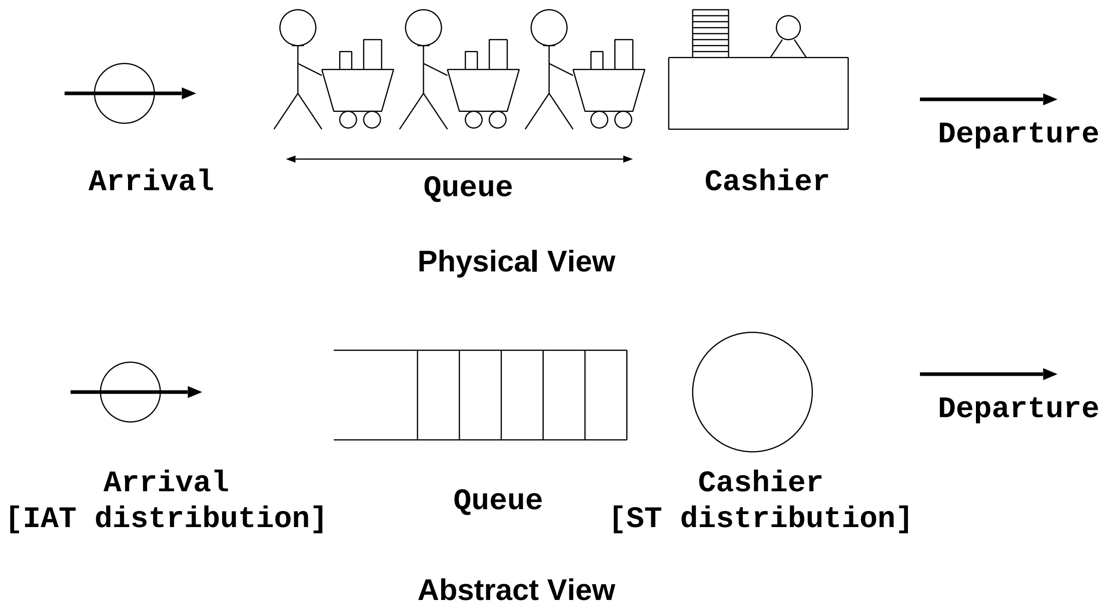
1. no output
2. with output

$\Rightarrow$ automatic (though inefficient) transformation

FSA without output

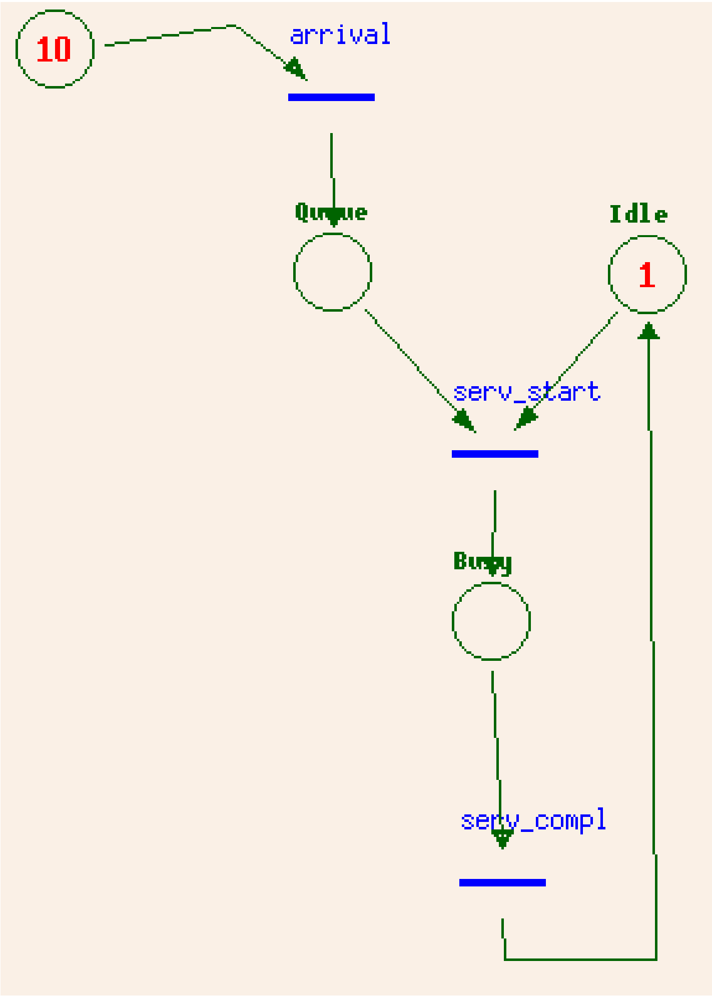
FSA with output

Petri net models for Queueing Systems

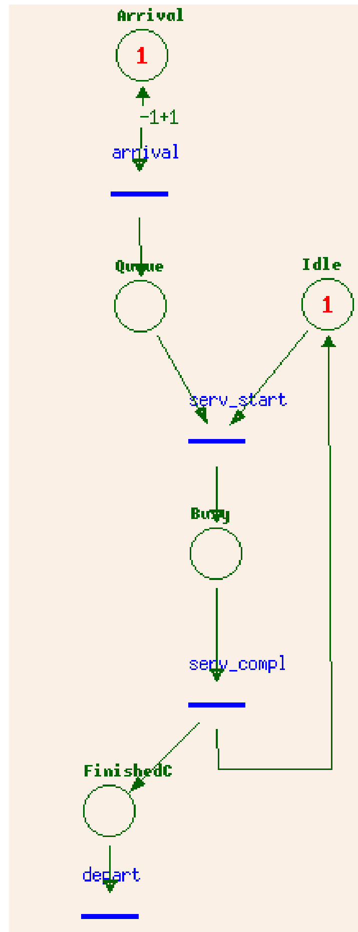


Capacity Constraints for Resource Conservation

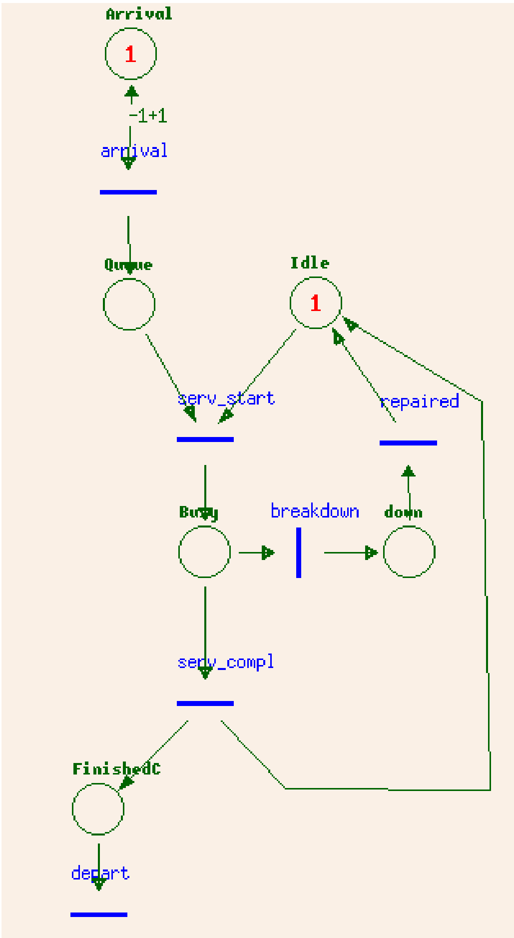
Simple Server/Queue Model



Model departure explicitly



Model Server Breakdown



Modular Composition: Communication Protocol

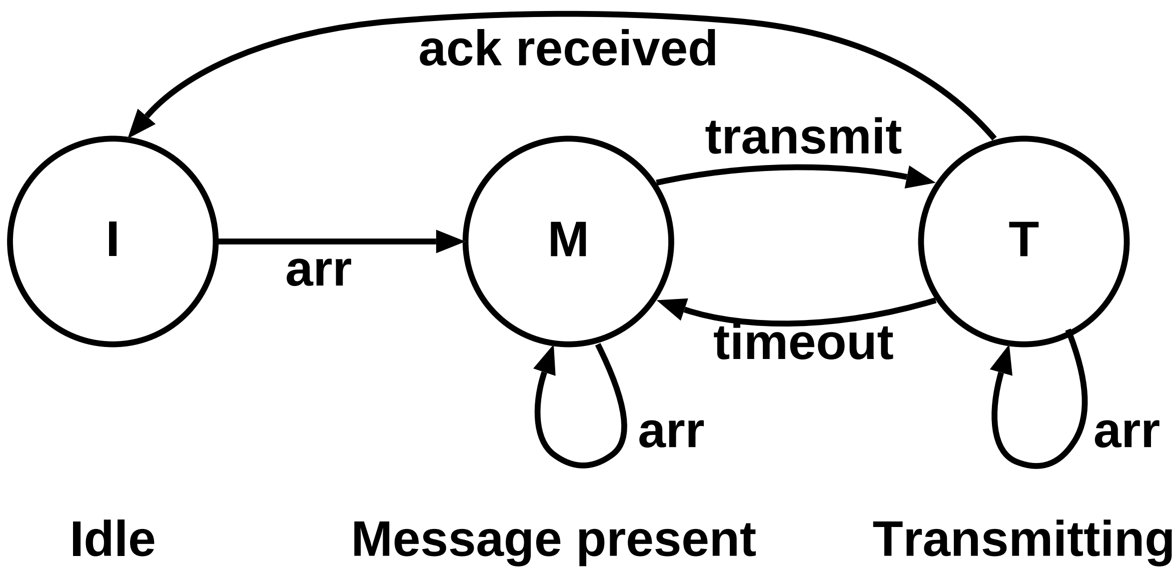
Build incrementally:

1. Single transmitter: FSA vs. Petri net
2. Two transmitters competing for channel

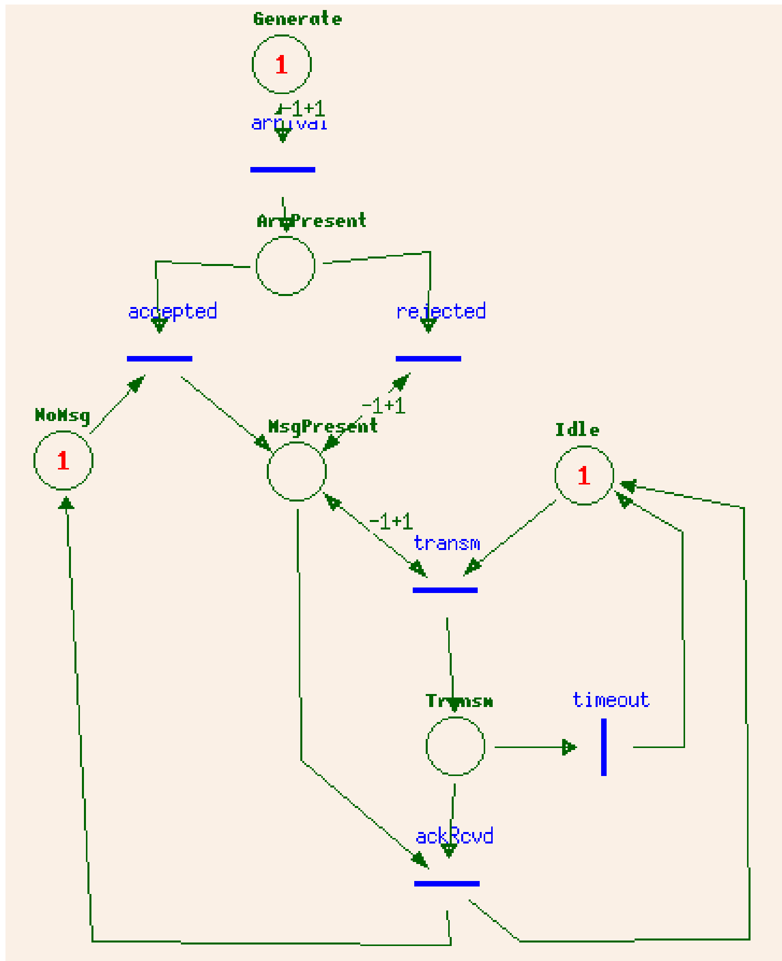
Pros/Cons of Petri net models (depends on goals !):

- Petri net is more complex than FSA for single transmitter
- More insight
- Incremental modelling
- Modular modelling
- Intuitive modelling of concurrency

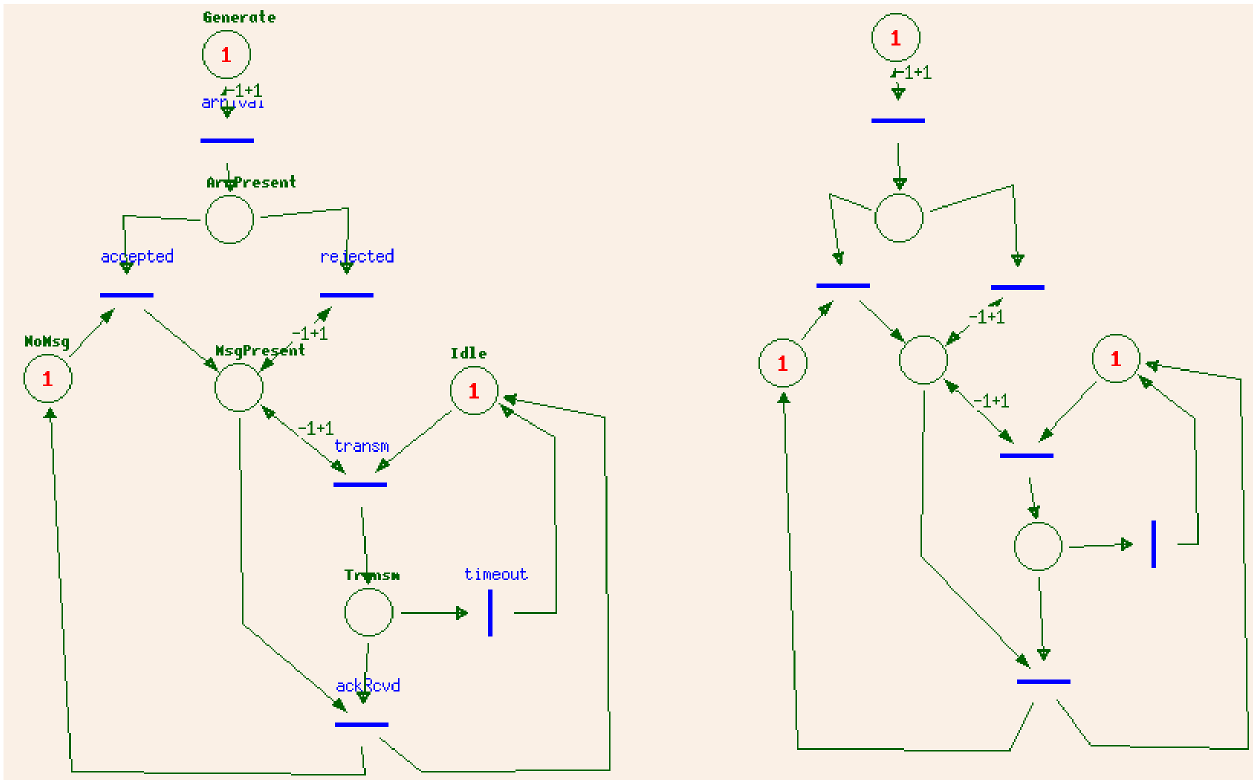
Single Transmitter FSA



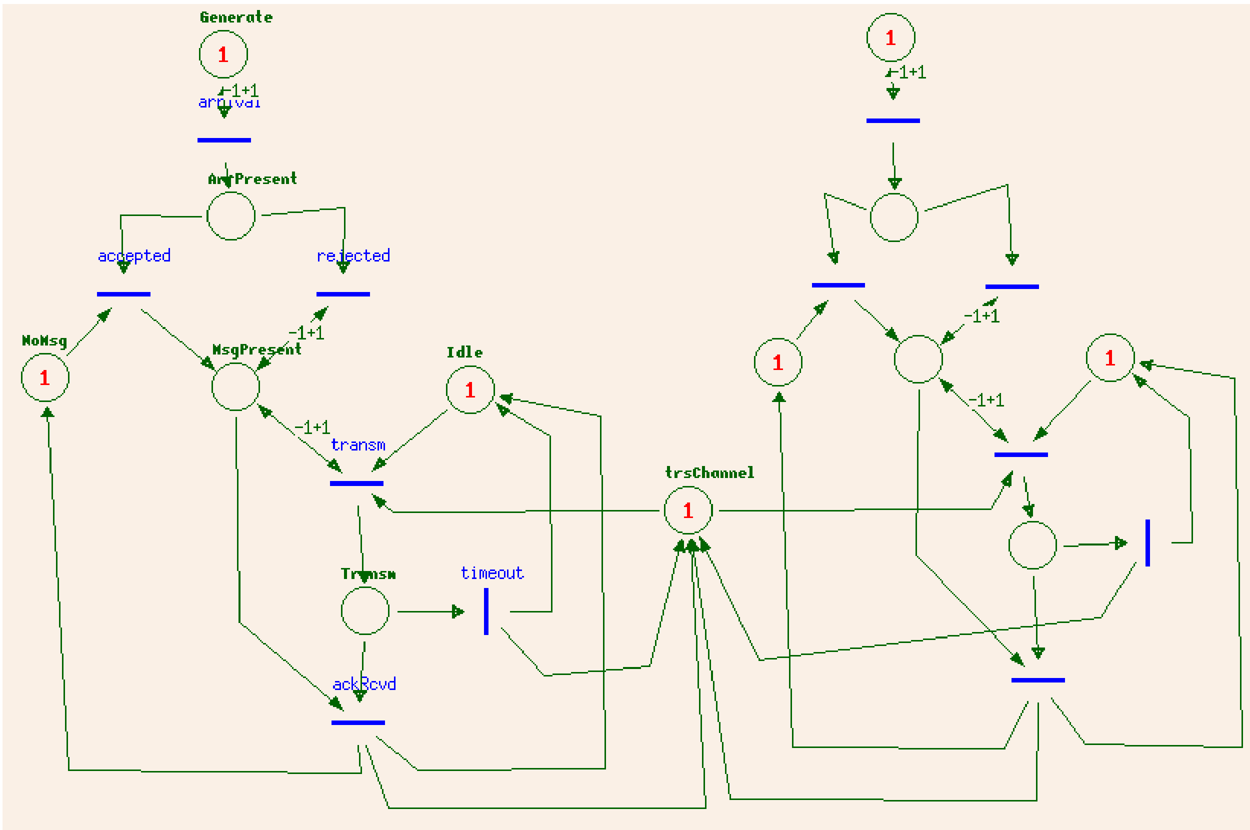
Single Transmitter Petri net



Concurrent, Non-interacting Transmitters



Concurrent, Interacting Transmitters



Analysis of Petri nets

Analysis of *logical* or *qualitative* behaviour.

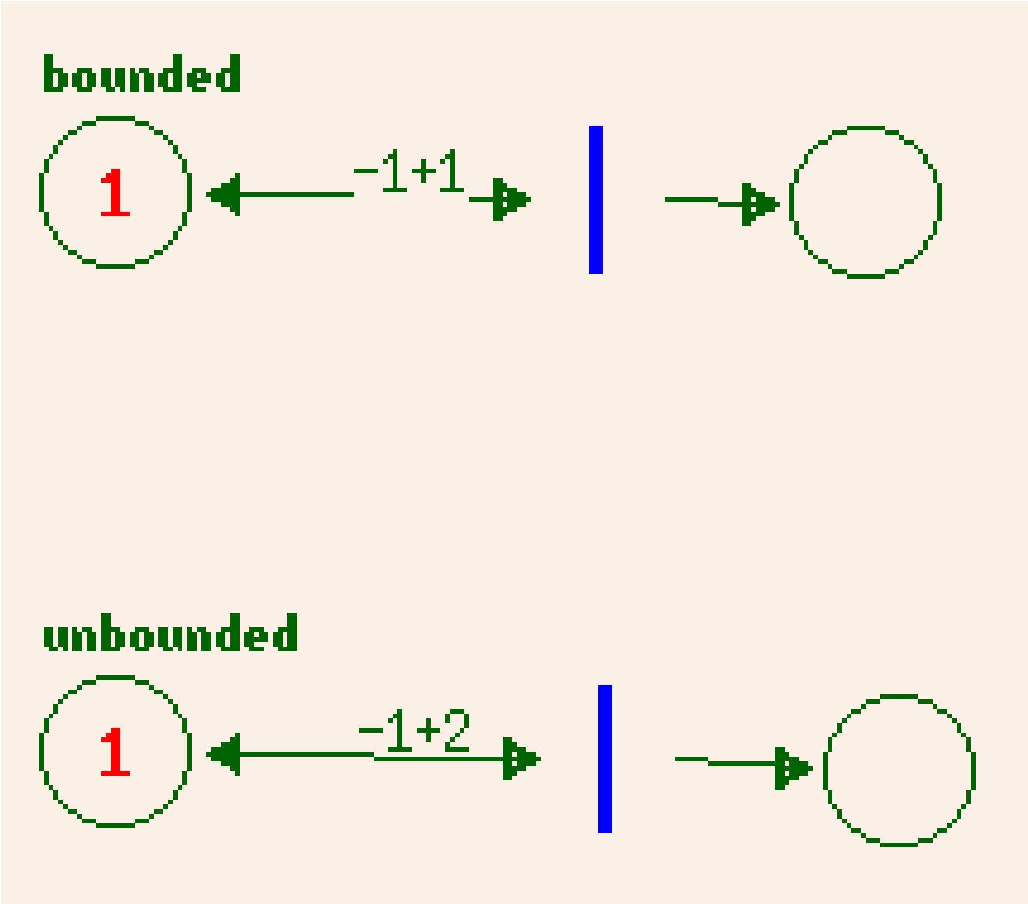
Resource sharing $\Rightarrow$ *fair* usage of resources:

- Boundedness
- Conservation
- Liveness and Deadlock
- State Reachability
- State Coverability
- Persistence
- Language Recognition

Boundedness

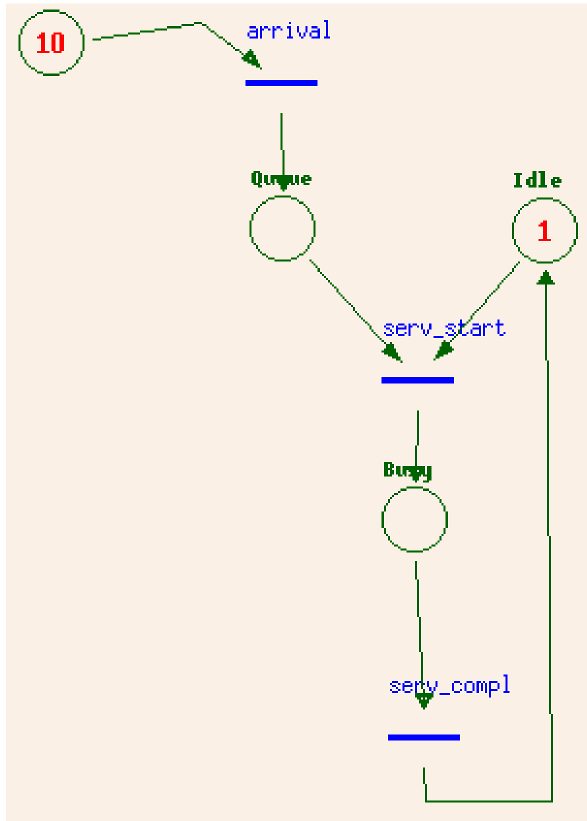
- Example: upper bound on number of customers in queue.
- Definition: A place $p_i \in P$ in a Petri net with initial state $\mathbf{x}_0$ is $k-$*bounded* or $k-$*safe* if
  $x(p_i) \leq k$ for all states in all possible sample paths.
- A $1-$bounded place is called *safe*.
- If a place is $k-$bounded for some k, the place is *bounded*.
- If all places are bounded, the Petri net is *bounded*.

Bounded vs. Unbounded



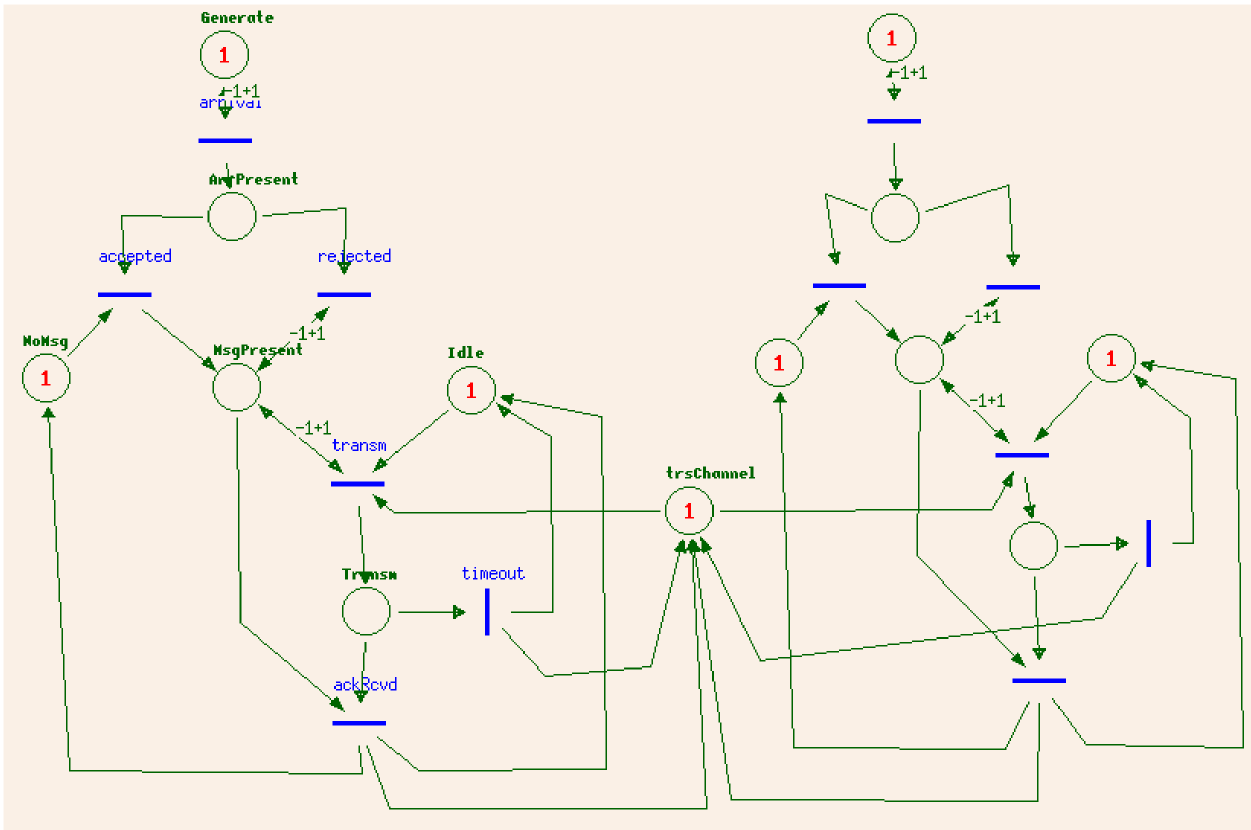
Conservation

Token represents *resource*, *process*, . . .



Sum $Busy + Idle$ tokens must be *constant* for all states in all sample paths

Conservation, weighted sum



2 Transm + Idle + trsChannel = constant

Conservation

A Petri net with initial state x_0 is *conservative with respect to* $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_n]$ if

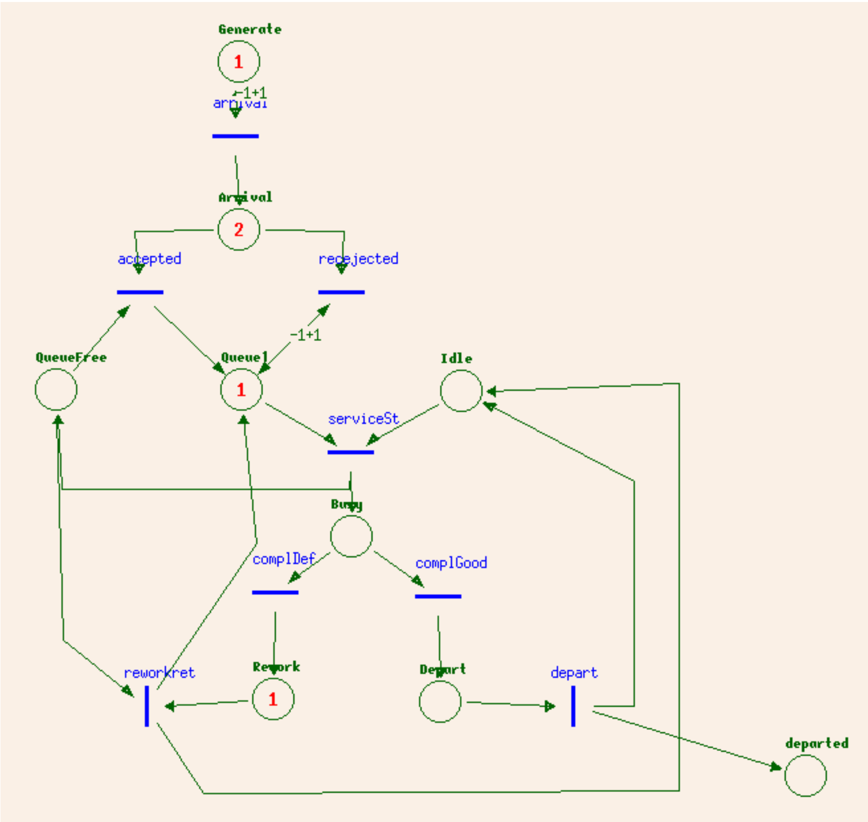
$$\Sigma_{i=1}^{n} \gamma_i x(p_i) = constant$$

for all states in all possible sample paths.

Liveness and Deadlock

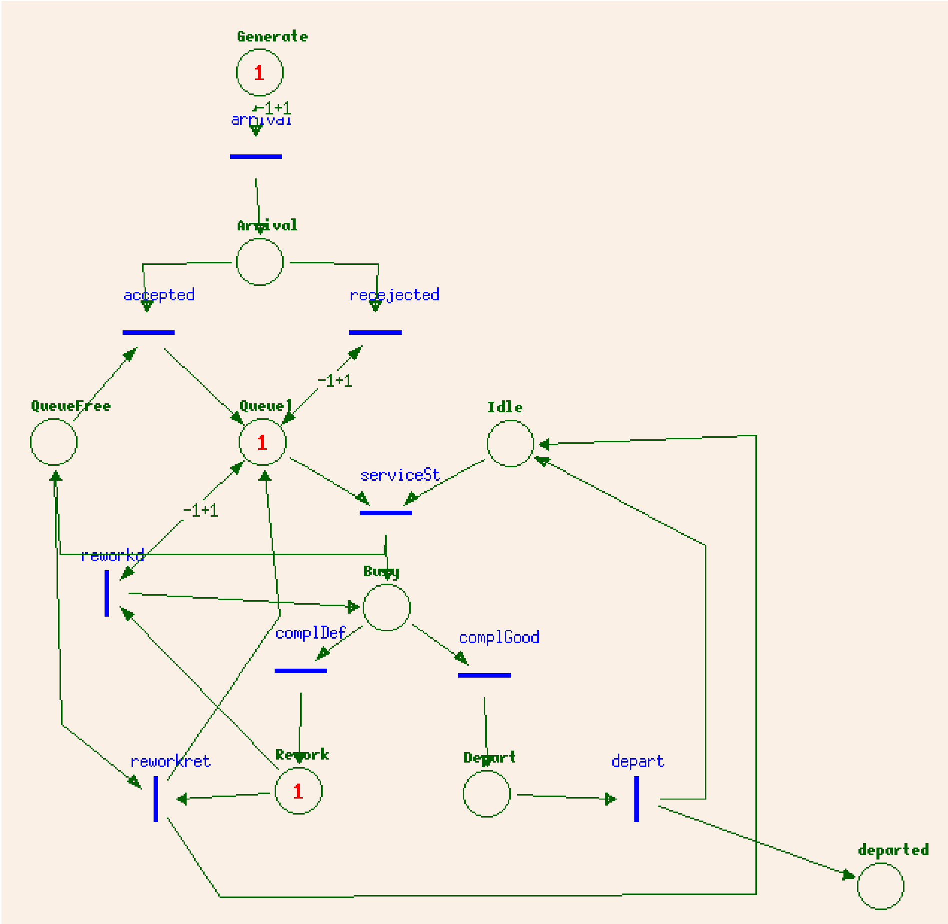
- Cyclic dependency $\Rightarrow$ wait indefinitely
- Deadlock
- Deadlock avoidance: avoid certain states in sample paths

Deadlock in Queueing system with Rework



$$[QueueFree, Queue1, Rework] = [0, 1, 1]$$

Deadlock resolved

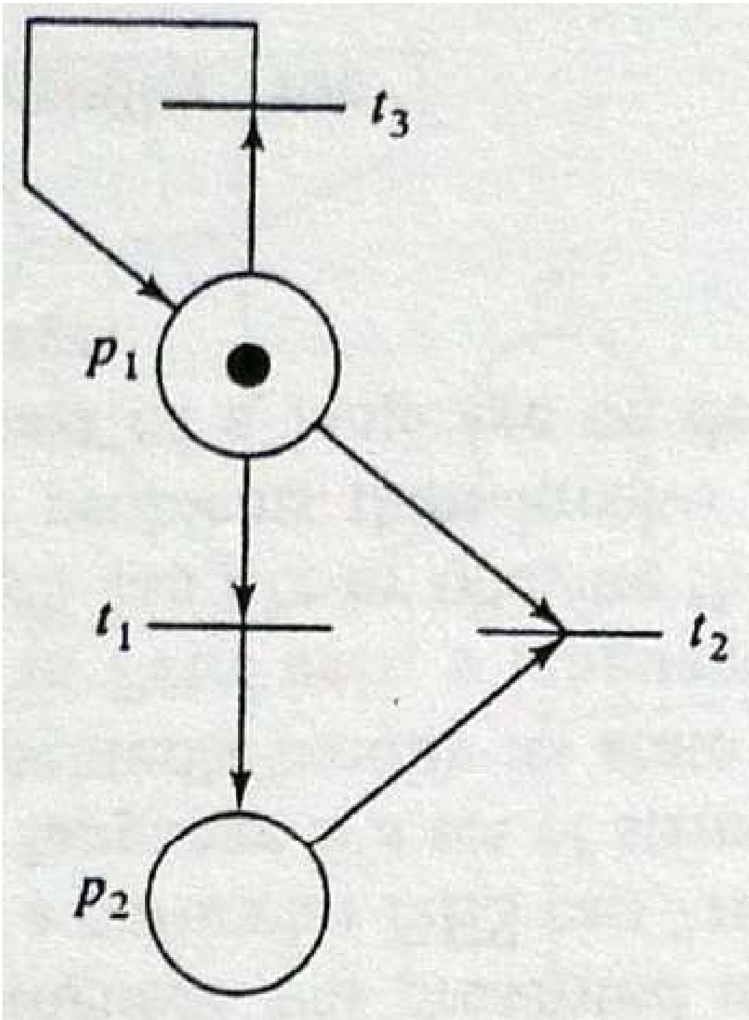


Liveness

Given initial state $\mathbf{x}_0$, a transition in a Petri net is:

- L0-live (dead): if the transition can never fire.
- L1-live: if there is some firing sequence from $\mathbf{x}_0$ such that the transition can fire at least once.
- L2-live: if the transition can fire at least k times for some given positive integer k.
- L3-live: if there exists some infinite firing sequence in which the transition appears infinitely often.
- L4-live: if the transition is L1-live for every possible state reached from $\mathbf{x}_0$.

Liveness example



- $t1$ is L1-live;
- $t2$ is dead;
- $t3$ is L3-live, not L4-live.

State Reachability

- A state $\mathbf{x}$ in a Petri net is *reachable* from a state $\mathbf{x}_0$ if there exists a sequence of transitions starting at $\mathbf{x}_0$ such that the state eventually becomes $\mathbf{x}$.
- Build/use reachability graph.
- Deadlock avoidance is a special case of reachability.

State Coverability

- In a Petri net with initial state $\mathbf{x}_0$, a state $\mathbf{y}$ is *coverable* if there exists a sequence of transitions starting at $\mathbf{x}_0$ such that the state eventually becomes $\mathbf{x}$ and $x(p_i) \geq y(p_i)$.
- Related to L1-liveness: *minimum number of tokens required* to enable a transition.

Persistence

- More than one transition enabled by the same set of conditions (choice, undeterminism).
- If one fires, does the other remain enabled ?
- A Petri net is *persistent* if, for any two enabled transitions, the firing of one cannot disable the other.
- Non-interruptedness (of multiple processes).

Language Recognition

Language defined by Petri net

$\equiv$

set of transition sequences which can fire

Coverability Notation

- Root node
- Terminal node
- Duplicate node

Coverability Notation

- Node *dominance*

$$\mathbf{x} = [x(p_1), x(p_2), \ldots, x(p_n)]$$

$$\mathbf{y} = [y(p_1), y(p_2), \ldots, y(p_n)]$$

$\mathbf{x} >_d \mathbf{y}$ ($\mathbf{x}$ dominates $\mathbf{y}$)if

1. $x(p_i) \geq y(p_i), \forall i \in \{1, \ldots, n\}$
2. $x(p_i) > y(p_i)$ for at least some $i \in \{1, \ldots, n\}$

- The symbol ω represents *infinity*

$$\mathbf{x} >_d \mathbf{y}$$

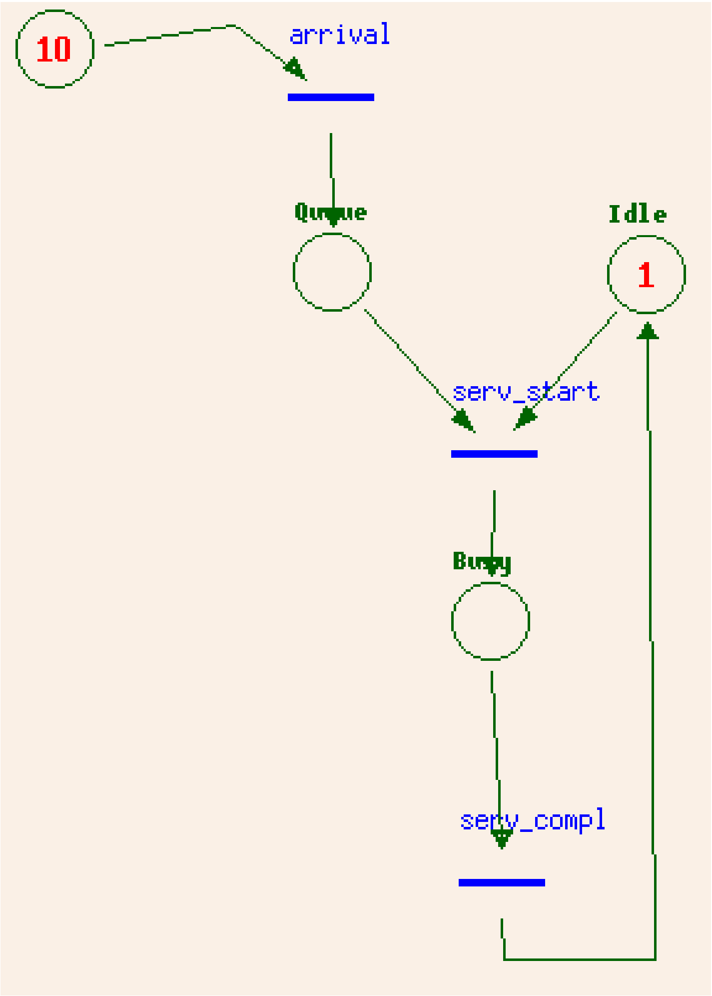
For all i such that $x(p_i) > y(p_i)$, replace $x(p_i)$ by ω

$$\omega + k = \omega = \omega - k$$

Coverability Tree Construction

1. Initialize $\mathbf{x} = \mathbf{x}_0$ (initial state)
2. Fore each new node $\mathbf{x}$,
   evaluate the transition function $f(\mathbf{x}, t_i)$ for all $t_j \in T$:
   (a) if $f(\mathbf{x}, t_j)$ is undefined for all $t_j \in T$, then $\mathbf{x}$ is a terminal node.
   (b) if $f(\mathbf{x}, t_j)$ is defined for some $t_j \in T$,
   create a new node $\mathbf{x}' = f(\mathbf{x}, t_j)$.
      i. if $x(p_i) = \omega$ for some p_i, set $x'(p_i) = \omega$.
      ii. If there exists a node $\mathbf{y}$ in the path from root node $\mathbf{x}_0$ (included) to $\mathbf{x}$ such that $\mathbf{x}' >_d \mathbf{y}$, set $x'(p_i) = \omega$ for all p_i such that $x'(p_i) > y(p_i)$
      iii. Otherwise, set $\mathbf{x}' = f(\mathbf{x}, t_j)$.
3. Stop if all new nodes are either *terminal* or *duplicate*

Coverability Tree Example: Cashier/Queue



Coverability Tree Example: Cashier/Queue

Applications of the Coverability Tree

- Boundedness: ω does not appear in coverability tree
- Bounded Petri net $\Rightarrow$ reachability graph
- Conservation: $\gamma_i = 0$ for ω positions
- Inverse problem: what are γ and C ?
- Coverability: inspect coverability tree
- Limitations: deadlock detection