

# Evolution kinetics and phase transitions of complex adaptive systems

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Motivated by the fact that a population of competing agents never set up a true stationary distribution, we propose a theory of evolution kinetics for complex adaptive systems. The formula derived for the survival probability is used to describe different phases in the population evolution when the prize-to-fine ratio as well as time changes. A kinetic phase diagram is obtained to show the phase boundaries for the self-segregating region, the intermediate region, and the clustering region. The kinetic evolutionary equations of strategy distribution are also established and used to give the root-mean-square separation of strategy distribution. It is revealed that the rootstock of the phase transitions is ascribed to the cooperation and competition among agents with different gene values for a limited resource.

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A common problem which is of central importance in social, economic, and biological sciences [1–5] is the evolving population in which individual members (agents) adapt their behavior according to past experience. The most interesting situation is that agents compete for a limited resource or to be in the minority. Considerable progress in the theoretical understanding of such systems has been made by studying a simple, yet realistic model of the minority game (MG) and its evolutionary version (EMG) [6–18]. The basic model of the EMG consists of  $N$  (odd) agents, each having some finite number of strategies. At each time step, each member has to choose one of the two kinds of actions, such as buying and selling the asset in a financial market. The payoff of the game is that after all agents have taken action independently, those who are in the minority group win and acquire a point. The agents make their decisions based on the common knowledge of the past record. The most interesting result [6,10] of the model is that some kind of cooperation and self-organization appears among the agents. It is possible that the agents can improve their performances by modifying their strategies through a genetic-algorithm-based crossover mechanism [12]. Another approach for the EMG is that each agent is assigned a single number or “genetic value”  $g$  [13]. Following a given  $m$ -bit record,  $g$  is the probability that the agent will choose the same outcome as that stored in the common knowledge, i.e., he will follow the current prediction, while  $1-g$  is the probability he will choose the opposite. The most remarkable conclusion deduced from this kind of EMG is that the population of competing agents tends to self-segregate into the opposing groups characterized by extreme behavior when the prize-to-fine ratio  $R$  equals unity, and in order to flourish in such situation an agent would behave in an extreme way with the gene value  $g=0$  or  $1$ . In order to explain this result, two analytic theories [14,15], both based on the stationary assumption, have been proposed. To consider the more realistic life situations, an extended EMG model has been explored, in which the prize-

to-fine ratio  $R$  takes a variety of different values [16]. It has been shown that sharp phase transitions exist in this model: “confusion” and “indecisiveness” take over in the time of depression with the prize-to-fine ratio  $R$  smaller than a critical value  $R_c$ . In this case, the cautious agents (characterized by  $g=1/2$ ) perform better than the extreme ones. That is, for  $R < R_c$ , agents tend to cluster around  $g=1/2$  rather than self-segregate into two opposing groups.

In fact, a population of competing agents never establishes a true stationary distribution. The winning probability of an agent is *oscillatorily time-dependent* [17]. In order to explain the global behavior of agents in the extended EMG, a random-walk (RW) model with a *time-dependent* jumping probability, corresponding to the winning probability, has been proposed [18]. In this model, a walker randomly walks on a one-dimensional lattice with discrete time  $t=0,1,2,\dots$ . The probability  $\tau(t)$  to step to the right is given by  $\tau(t)=\frac{1}{2}-\varepsilon-(-1)^t A$ . The biased value  $\varepsilon$  is often positive and  $\ll 1/2$ . For definiteness,  $A$  is taken as positive and represents the amplitude of temporal oscillations. To let the probability be meaningful, it is required that  $A < \frac{1}{2}-\varepsilon$ . It is noted that this theory is based on the stationary assumption by adopting  $\varepsilon(g)=(\varepsilon_0/\sqrt{N})g(1-g)$  [14]. Therefore, phase transitions are dependent on the bias and will disappear if  $N \rightarrow \infty$ . Moreover, there is no time-dependent phase transition for any given prize-to-fine ratio  $R$ . These seem not to be found in real circumstances and are also in contradiction to numerical simulations.

The aim of this paper is to study the evolution kinetics of a system composed of adaptive competing agents, which does not depend on any stationary assumption. We start to investigate the evolution of individual agents by RW in one dimension. Due to the well-known fact that the winning probability of an agent may experience periodic oscillations [16,17], we can differentiate two time units used for convenience. One is the time step for the agent to make a decision, and the other is the oscillating half-period for winning or losing sequentially. Generally the latter, denoted by  $T/2$ , includes more than one time step. In our approach, a time-dependent jumping probability density is defined as

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$$w(x, t; x', t-1) = \left[ \frac{1}{2} - (-1)^t A \right] \delta(x, x' + Rl) + \left[ \frac{1}{2} + (-1)^t A \right] \delta(x, x' - l), \quad (1)$$

which describes a process in which a walker jumps from site  $x'$  at time  $t-1$  to site  $x$  at time  $t$ . Here the oscillating half-period  $T/2$  is taken as the time unit, the parameter  $R$  is the ratio of two distances jumping to the right and to the left (i.e., the prize-to-fine ratio in the EMG), and  $l$  defines the jumping distance to the left in a half-period. In the reduced units,  $l=T/2$ . Furthermore,  $p(x, t)$  is used to denote the spatio-temporal probability density for the random walkers at position  $x$  at time  $t$ . Its evolution obeys the general master equation

$$p(x, t) = \int w(x, t; x', t-1) p(x', t-1) dx' = \left[ \frac{1}{2} - (-1)^t A \right] p(x - Rl, t-1) + \left[ \frac{1}{2} + (-1)^t A \right] \times p(x + l, t-1). \quad (2)$$

Based on Eq. (2), some important quantities are derived by using the characteristic function method [19]. The first-order moment and the second-order central moment are given by

$$\sigma_1 \equiv x_0 + \langle x \rangle_t = x_0 - \frac{1}{2}(1-R)lt - Al(1+R)\sin^2 \frac{\pi t}{2} \quad (3)$$

and

$$\sigma_2 \equiv \langle (x - \sigma_1)^2 \rangle_t = \left( \frac{1}{4} - A^2 \right) (1+R)^2 l^2 t - 2(1-R^2)l^2 A \sin^2 \frac{\pi t}{2}, \quad (4)$$

where  $x_0$  is the initial position of the walker and is assumed larger than zero in general. According to the central-limit theorem, for a long enough time  $t$ , the probability density  $p(x, t)$  can be approximated by the Gaussian distribution with the expectation value  $\sigma_1$  and the square standard deviation  $\sigma_2$ . It tells us, from Eqs. (3) and (4), that the distribution center moves with an oscillating drift speed towards the left and the distribution widens oscillatorily with time.

In the EMG, an agent always tries to locate in the minor group, so there is a tendency to form a probability flow from high density to low density. This makes the diffusion approximation a valid approach for further dealing with our problem. In this approximation [19], the evolution of  $p(x, t)$  after a long time can be described by the variable coefficient diffusion equation

$$\frac{\partial p(x, t)}{\partial t} = D(t) \frac{\partial^2 p(x, t)}{\partial x^2} - v(t) \frac{\partial p(x, t)}{\partial x}, \quad (5)$$

where the diffusion coefficient and the drift velocity are related to Eqs. (3) and (4) by  $\sigma_2 = 2 \int_0^t D(t) dt$  and  $\langle x \rangle_t = \int_0^t v(t) dt$ , respectively. When time  $t$  is long enough, the asymptotic solution of Eq. (5), satisfying the initial condition

$p(x, 0) = \delta(x - x_0)$  and the absorbing boundary condition  $p(0, t) = 0$ , can be expressed as

$$p(x, t; x_0, 0) = \frac{1}{4\sqrt{2\pi\sigma_2}} \left\{ \exp \left[ -\frac{(x - x_0 + \langle x \rangle_t)^2}{2\sigma_2} \right] + \exp \left[ -\frac{(x - x_0 - \langle x \rangle_t)^2}{2\sigma_2} \right] - \exp \left[ -\frac{(x + x_0 + \langle x \rangle_t)^2}{2\sigma_2} \right] - \exp \left[ -\frac{(x + x_0 - \langle x \rangle_t)^2}{2\sigma_2} \right] \right\}. \quad (6)$$

From this equation, we can obtain an analytic expression for the survival probability,

$$S(t|x_0) = \int_0^\infty p(x, t; x_0, 0) dx = \frac{1}{2} \left[ \operatorname{erf} \left( \frac{x_0 + \langle x \rangle_t}{\sqrt{2\sigma_2}} \right) + \operatorname{erf} \left( \frac{x_0 - \langle x \rangle_t}{\sqrt{2\sigma_2}} \right) \right], \quad (7)$$

where  $\operatorname{erf}(x)$  is the error function. Equation (7) is the central result of this paper, from which a lot of relevant information regarding the evolving population of a complex adaptive system can be extracted.

From Eq. (7) we can investigate the evolution kinetics of a competing and adaptive agent system by considering an agent's survival probability depending on his gene value and on time. It is shown that the amplitude of the oscillating winning probability [17,18] can be expressed by  $A = a(R)|g - 1/2|$  and its period  $T$  also varies with the prize-to-fine ratio  $R$ . According to the previous and also our numerical simulations, we find it is appropriate to adopt  $a(R) = 1/[1 + \exp(1-R)]$ , which gives  $A = 0.491$  when  $R = 0.8$ , and  $A = 0.25$  when  $R = 1$ , for  $g = 0$  or  $1$ ; and  $T(R) = 10[1 + \exp(1-R)]^2$ , which corresponds to  $T \approx 10$  time steps when  $R = 0.8$  and  $T = 40$  time steps when  $R = 1.0$ . Correspondingly, from  $l = T/2$ , the jumping distance of the walker to the left is 5 and 20, respectively. It should be noted that so far the prize-to-fine ratio dependence of the oscillating amplitude and period can only be shown by numerical simulations. The two expressions here for  $a(R)$  and  $T(R)$  are obtained by fitting the numerical results and they are consistent with Figs. 1, 2, and 4 in [17] qualitatively. In fact, we have also written  $a(R)$  and  $T(R)$  in other functional form, but found there are no obvious deviations in the phase diagram. Substituting these two expressions for the amplitude and period into Eqs. (3), (4), and (7), we can analyze the behavior of agents with different gene values. For a given initial position  $x_0$ , we can plot the survival probability  $S(t, R)$  with respect to gene value  $g$  from 0 to 1 for a different prize-to-fine ratio  $R$  at a given time  $t$  or at different times  $t$  for a given prize-to-fine ratio  $R$ . The former is the same result obtained by Hod with a different approach [18], but the latter is totally new. Generally, we can find three kinds of distributions, or three phases, shown in Fig. 1; the parameter  $x_0$  is chosen to be 4. The U-shaped curve represents the self-segregation, which corresponds to

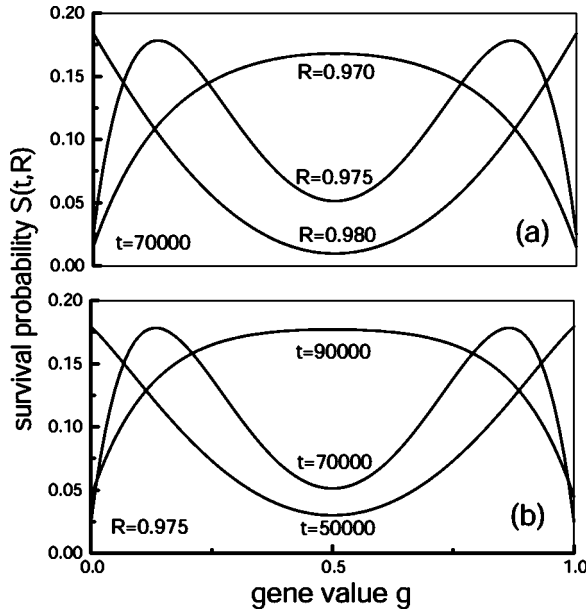


FIG. 1. Survival probability vs gene value. (a) At a fixed time for three different prize-to-fine ratios; (b) for a fixed prize-to-fine ratio at three different times. The graphs are rescaled for convenience and the time step is chosen as the time unit.

the case in which there are less *cautious* agents (characterized by  $g \approx 1/2$ ) than *extreme* agents (characterized by  $g \approx 0$  or  $1$ ); the  $\cap$ -shaped curve corresponds to the case opposite to the first one; and the M-shaped curve is the intermediate phase.

From the analytic results obtained above, the evolutionary behavior of complex adaptive systems can be interpreted as follows. On the one hand, for the agents near an extremum characterized by  $g \approx 0$  or  $1$ ,  $A \approx A_{\max}(R)$ , while for the cautious agents characterized by  $g \approx 1/2$ ,  $A \approx 0$ . Therefore, it can be seen clearly from Eq. (4) that in the case of any value of  $R$ , the scores of the cautious agents disperse more quickly than those of the extreme agents. On the other hand, the distribution center of scores for all agents drifts toward to the absorbing boundary with the same drift speed  $v(t) = d\sigma_1/dt$  for  $R < 1$  and dangles about  $x_0$  for  $R = 1$ . As a result, for  $R = 1$ , the survival probability of an extreme agent is always larger than that of a cautious agent, and the survival probability distribution of the whole system shows a  $\cup$  shape, i.e., a self-segregating phase. For  $R < 1$ , the scores of agents decrease oscillatorily. Once their scores fall below some value, they will eventually perish. Thus, at the beginning, the survival probability of an extreme agent is larger than that of a cautious agent, and the survival probability distribution of the whole system shows a  $\cup$  shape, i.e., a self-segregating phase; in the late period, most of the extreme agents' scores have fallen below critical value  $d$ , but some of the cautious agents have not and the survival probability distribution of the whole system shows a  $\cap$  shape, i.e., a clustering phase. In the intermediate period, the survival probability distribution of the whole system shows an M shape, i.e., an intermediate phase. In one word, the phase transitions are related to the presence of a drift speed for the distribution center of the scores of agents, and also to the difference of oscillating

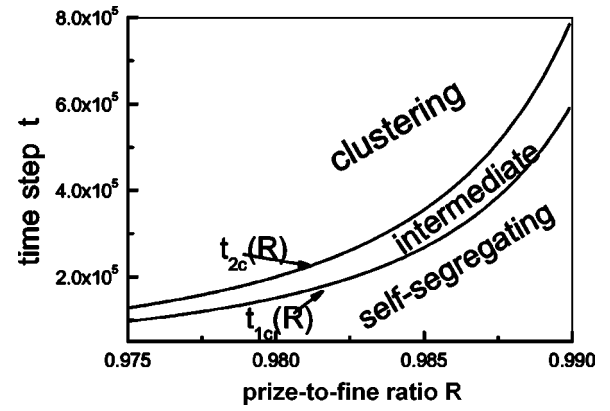


FIG. 2. Kinetic phase diagram.

amplitudes of the winning probabilities of agents with different gene values.

Furthermore, we can determine two critical times to characterize the kinetic phase transitions, i.e., from the self-segregating phase ( $\cup$  phase) to the intermediate phase (M phase) and from the intermediate phase to the clustering phase ( $\cap$  phase). Both critical times can be determined by letting the first derivative and the second derivative of the survival probability  $S$  with respect to the gene value  $g$  be equal to zero. For the  $\cup$ -M phase transition, the first derivative is taken at  $g = 0$  (or  $1$ ) and for the M- $\cap$  phase transition, the second derivative is taken at  $g = 1/2$ . After a little mathematics, the critical times are determined. They read

$$t_{1c} = \frac{2x_0}{1-R} \coth \left[ \frac{2x_0(1-R)}{(1-a^2)(1+R)^2 l} \right] \quad (8)$$

and

$$t_{2c} = \frac{2x_0}{1-R} \coth \left[ \frac{2x_0(1-R)}{(1+R)^2 l} \right]. \quad (9)$$

It is noted that in deriving these two expressions, the oscillating terms in Eqs. (3), (4), and (7) are omitted for simplicity, because they are trivial for the long-time effect. Equations (8) and (9) can be used to give a kinetic phase diagram which describes what kind of state the system is in at a given time for a given prize-to-fine ratio. Plotting  $t_{1c}$  and  $t_{2c}$  with respect to  $R$  for the given parameter  $x_0$ , we can find that the curves  $t_{1c}(R)$  and  $t_{2c}(R)$  divide the  $(t, R)$  plane into three regions as shown in Fig. 2, in which the upper, middle, and bottom regions correspond to the clustering, intermediate, and self-segregating phases, respectively. From Eqs. (8) and (9) and Fig. 2, we can draw some interesting conclusions. (i) To express  $t$  as the functions of  $R$ , there is a free parameter  $x_0$  as the initial value for an agent to enter in EMG. Its variation will not affect the analytic property of the phase boundaries  $t_{1c}$  and  $t_{2c}$ , so the phase diagram is stable under the perturbation by  $x_0$ , although the system itself is nonstationary. (ii) If  $R \rightarrow 1$ ,  $t_{1c}$  and  $t_{2c}$  tend to infinity and it is impossible for a phase transition to appear. That is to say, the population of competing agents with similar capability and knowledge will always tend to self-segregate into the opposite groups characterized by the extreme behavior, which is in agreement

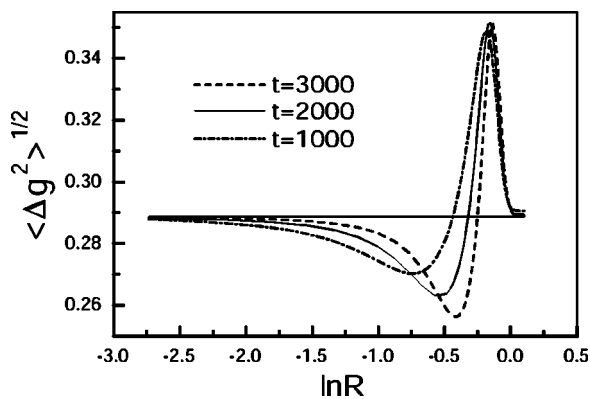


FIG. 3. Root-mean-square (rms) separation of strategies vs prize-to-fine ratio at  $t=1000$ ,  $2000$ , and  $3000$  time steps, respectively, for  $x_0=4.0$ . The horizontal line represents the rms separation, which equals  $1/\sqrt{12}$ , for the uniform  $P(g)$  distribution.

with the result presented by Johnson *et al.* [13]. (iii) At any long enough time, there are generally three phases which correspond to the regions with different prize-to-fine ratios  $R$  and are in agreement with the result presented by Hod [16,18]. (iv) For any  $R < 1$ , there will always be phase transitions as time changes from the self-segregating phase to the intermediate phase and from the intermediate phase to the clustering phase.

Equation (7) is also useful to elucidate the kinetic evolutionary characteristics of the strategy distribution of agents,  $P(g, t)$ , which could be expressed as

$$P(g, t) = \sum_{i=0}^t p_i(g, t). \quad (10)$$

Here  $p_i(g, t)$  is taken as the fraction of agents who have survived after  $i$  time steps since they enter the game. Therefore,

$$p_0(g, t) = \int_0^1 dg \sum_{i=0}^{t-1} p_i(g, t-1) [S(g, i) - S(g, i+1)] \quad (11)$$

and

$$p_i(g, t) = p_{i-1}(g, t-1) [1 - S(g, t-1) + S(g, t)], \quad i = 1, \dots, t. \quad (12)$$

Figure 3 displays the root-mean-square (rms) separation of strategies  $\langle (\Delta g)^2 \rangle^{1/2}$  as the function of prize-to-fine ratio  $R$

obtained by numerically solving Eqs. (10)–(12), which agrees qualitatively with the result presented in Fig. 2 in Ref. [16] by numeral simulations.

In summary, based on the fact that a population of competing agents never establishes a true stationary distribution, a theory has been proposed to study the evolution kinetics for complex adaptive systems according to the characteristic function method and the variable coefficient diffusion equation. In the presence of an absorbing boundary, an analytic expression for the survival probability was derived and used to show the kinetic phase transitions from the self-segregating phase to the intermediate phase and from the intermediate phase to the clustering phase when the prize-to-fine ratio as well as time changes. Two formulas for the kinetic phase boundaries were obtained and then a kinetic phase diagram was plotted. In addition, the kinetic evolution of strategy distribution was also analytically treated and the results are in agreement with the numerical simulations. It has been made clear that the rootstock of phase transitions in complex adaptive systems can be ascribed to the cooperation and competition among agents with different gene values and a limited resource characterized by the prize-to-fine ratio. For any value of the fine-to-prize ratio less than 1, the population distribution of the system will always experience three different phases. At the beginning, the survival probability of an extreme agent is larger than that of a cautious agent, and the survival probability distribution of the whole system shows a  $\cup$  shape, corresponding to the self-segregating phase; the extreme agents behave better than the cautious agents. In the late period, most of the extreme agents' scores have fallen below a critical value, while some of the cautious agents' scores have begun to rise above this value. Then the survival probability distribution of the whole system shows a  $\cap$  shape, corresponding to the clustering phase; the cautious agents will behave better than the extreme agents. Between these two extreme periods, the survival probability distribution of the whole system shows an M shape related to the intermediate phase. Our theory definitely confirms the fact that a population of competing agents never sets up a true stationary state.

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