# CONSTITUTIVE HYBRID PROCESSES

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## Abstract

This paper studies constitutive relations of physical processes with time-scale abstractions by giving a process algebraic foundation for hybrid bond graph models. For a number of standard physical elements, it is shown how discontinuous constitutive relations that describe the behavior of hybrid bond graph models can be derived from existing continuous constitutive relations. These discontinuous and continuous descriptions are then combined using the hybrid process algebra HyPA, which may be used to reason symbolically about both the continuous behavior, and the mode switching that results from the time-scale abstraction.

## 1 Introduction

A physical system is often modeled by aggregating constitutive relations on pertinent phenomena. At a macroscopic level, the corresponding behavior can be considered continuous, and the constitutive relations can be stated as differential equations possibly extended by algebraic constraints. When part of the continuous behavior occurs very fast it may be beneficial to describe this behavior as discontinuous. In this case, the constitutive relations may be extended with explicit re-initialization constraints (e.g. for a bouncing ball that reverses its velocity after a collision, $v^+ = -v^-$).

In this paper, the constitutive relations of a set of fundamental phenomena in physical modeling are described using the hybrid process algebra HyPA [Cuijpers and Reniers]. This algebra allows for the description of combinations of continuous and discontinuous behavior as one, hybrid, process. The algebra builds on bond graphs [Paynter 1961], a formalism that unifies different domains in physics, e.g., electronics, hydraulics, and mechanics. In particular, bond graphs that are extended with elements to describe discontinuous behavior [Breedveld 1996, Mosterman et al.1998, Stromberg et al. 1993] are used. This paper, can therefore also be considered an attempt to give a process algebraic semantics to such *hybrid* bond graphs.

This paper assumes familiarity with the bond graph formalism (see, e.g., [Karnopp et al. 1990]). In Section 2, a subset of the hybrid process algebra HyPA [Cuijpers and Reniers] is discussed briefly, with a focus on the constructs for describing physical behavior. In Section 3, hybrid constitutive relations of a number of bond graph elements are given as hy-

brid processes. In Section 4, a bond graph model of two colliding masses and its process algebraic representation are presented. Section 5 gives the conclusions of this work. A more detailed version of this paper [Cuijpers et al. 2004] extends the theory by including transformer and gyrator bond graph elements.

## 2 Hybrid process algebra

The hybrid process algebra HyPA (see [Cuijpers and Reniers] for a complete introduction, and a comparison with other hybrid formalisms), allows modeling of both continuous and discontinuous behavior. The full algebra also allows modeling of software components, through abstract computational actions in the style of the Algebra of Communicating Processes (i.e. ACP, see [Baeten and Weijland 1990]). However, software components are beyond the scope of this paper. Only the following signature of constants and function symbols is discussed, of which only $\oplus$ and $\|$ originate from ACP:

1. flow clauses $(V|\, P_{\mathrm{f}})$,
2. process re-initializations $[V|\, P_{\mathrm{r}}] \gg \_$,
3. alternative composition $\_ \oplus \_$,
4. disrupt $\_ \blacktriangleright \_$, and
5. parallel composition $\_ \| \_$.

The binding order of these is: $\blacktriangleright$, $\gg$, $\|$, $\oplus$, where $\oplus$ binds weakest, and $\blacktriangleright$ binds strongest.

Continuous and discontinuous behavior are described using predicates over model variables $\mathcal{V}_{\mathrm{m}}$. In this paper, continuous behavior is described using flow predicates $\mathcal{P}_{\mathrm{f}}$ on the model variables $\mathcal{V}_{\mathrm{m}}$ and their time derivatives $\dot{\mathcal{V}}_{\mathrm{m}} = \{\dot{x} \mid x \in \mathcal{V}_{\mathrm{m}}\}$. Discontinuous behavior is described using re-initialization predicates $\mathcal{P}_{\mathrm{r}}$ on model variables with a '-' superscript, $\mathcal{V}_{\mathrm{m}}^{-}$, to denote conditions that hold at the start of a re-initialization, and on model variables with a '+' superscript, $\mathcal{V}_{\mathrm{m}}^{+}$, to denote conditions that hold at the end of a re-initialization. As an abbreviation, primed versions $\mathcal{V}'_{\mathrm{m}}$ of the model variables are used to denote $x' = x^{+} - x^{-}$.

A flow clause is an atomic process that models the repeated execution of physical behavior. It is denoted as a pair $(V|\, P_{\mathrm{f}})$ of a set of model variables $V \subseteq \mathcal{V}_{\mathrm{m}}$, signifying which variables are not allowed to jump in between flows, and a flow predicate $P_{\mathrm{f}}$ modeling which flow behavior can be executed by the clause. A process re-initialization $[V|\, P_{\mathrm{r}}] \gg p$ models the behavior of a process $p$ where the model variables are first submitted to a discontinuous change. This change is specified by the set of model variables $V \subseteq \mathcal{V}_{\mathrm{m}}$ and the re-initialization predicate $P_{\mathrm{r}}$. In the case of process re-initializations, the set $V$ models which variables are allowed to change. In this paper, the changes that take place using re-initializations, are always modeled explicitly. Therefore, $V = \mathcal{V}_{\mathrm{m}}$ in all flow-clauses and re-initialization clauses that are used, and consequently, $(P_{\mathrm{f}})$ is used to denote $(\mathcal{V}_{\mathrm{m}}|\, P_{\mathrm{f}})$, and $[P_{\mathrm{r}}]$ is used to denote $[\mathcal{V}_{\mathrm{m}}|\, P_{\mathrm{r}}]$.

The alternative composition $p \oplus q$ models a (non-deterministic) choice between the processes $p$ and $q$. The disrupt $p \blacktriangleright q$ models that process $q$ may take over execution from process $p$ at any moment. In other words, it models a mode switch from $p$ to $q$. The parallel composition $p \| q$ models concurrent execution of $p$ and $q$. For the restricted version of HyPA that is used in this paper, the intuition behind parallel composition is simply that processes synchronize the execution of their continuous and discontinuous behavior, by simultaneously solving all parallel flow-predicates or re-initialization predicates.

Finally, more complex processes can be defined using recursive specifications $X \approx p$, where $X$ is a process variable and $p$ is a term possibly containing $X$ and other process variables. Recursion is a powerful mechanism to, e.g., express repetition in a process. Note that $\approx$ is used to denote equivalence of processes, while $=$ is used in flow and re-initialization predicates.

Using hybrid process algebra, a constitutive process for each element of a bond graph can be defined that describes its continuous and discontinuous behavior. This process takes the form:

$$X \approx \begin{array}{c} [\ \mathcal{V}_{\mathrm{m}} \mid P_{\mathrm{r}1}\ ] \gg \left(\ \mathcal{V}_{\mathrm{m}} \mid P_{\mathrm{f}1}\ \right) \blacktriangleright X \\ \oplus \ldots \oplus \\ [\ \mathcal{V}_{\mathrm{m}} \mid P_{\mathrm{r}n}\ ] \gg \left(\ \mathcal{V}_{\mathrm{m}} \mid P_{\mathrm{f}n}\ \right) \blacktriangleright X \end{array}$$

This models a repetitive choice on the execution of discontinuities followed by continuous behavior. A complete system is modeled as a parallel composition of constitutive processes, i.e.

$$S \approx X_0 \parallel \ldots \parallel X_m.$$

# 3 Constitutive processes

A bond graph can be interpreted as a graphical representation of the constitutive relations that describe a system. It consists of elements, junctions and bonds. The elements represent physical concepts of energy storage in the system, the junctions represent the conservation laws that play a role in the system, and the bonds, by connecting elements and junctions, describe the way in which energy flows through the system.

In this section, a hybrid interpretation to bonds, junctions and many of the standard elements of bond graph theory is given, by representing them as a constitutive process. This is also done for the hybrid elements defined in [Breedveld 1996, Mosterman et al.1998], that describe a changing connection structure between the elements of a bond graph. The discontinuous constitutive relations of a bond graph element are derived from its continuous constitutive relations, by substitution and integration.

**Bond** The main variables that play a role in a bond graph description, are effort $e$, flow $f$, stored energy $E$, generalized momentum $p$, and generalized displacement $q$. For some elements, these last two variables, $p$ and $q$, do not have a real physical interpretation, but are considered auxiliary variables that do not engage in discontinuous behavior.

The standard relations between variables are reflected in the constitutive process for a bond. Often, these relations are left implicit, for brevity of the presentation. Note, that a bond does not restrict the behavior during discontinuities in any way, which is reflected in the re-initialization clause [*true*].

$$\mathrm{B} \approx [\ \mathit{true}\ ] \gg \begin{pmatrix} \dot{E} = e \cdot f \\ \dot{p} = e \\ \dot{q} = f \end{pmatrix} \blacktriangleright \mathrm{B}$$

**Inductance and capacitance** An inductance is an element with a continuous constitutive relation $f = \mathrm{I}(p)$. This models the physical concept that energy is stored in the from of generalized momentum. Dually, a capacitance is an element with a continuous constitutive relation $e = \mathrm{C}(q)$. This models that energy is stored in the form of generalized displacement. For intrinsic stability of these elements, it is often required that the functions I and C satisfy $\frac{\partial \mathrm{I}(p)}{\partial p} > 0$ and $\frac{\partial \mathrm{C}(q)}{\partial q} > 0$, respectively, but we do not use this in the remainder of the paper.

For the stored energy in an inductance, the continuous relation $\dot{E} = e \cdot f = \dot{p} \cdot \mathrm{I}(p)$ can be derived. The substitution rule from differential calculus, and integration over the trajectory of $p$, then lead to $E' = \int_{p^-}^{p^+} \mathrm{I}(x)\ dx$, which serves as a discontinuous constitutive relation for the change in stored energy as a function of the change in generalized momentum. Note, that this derivation is only correct if $\mathrm{I}(p)$ does not depend (significantly) on other model variables. If $\mathrm{I}(p)$ does depend on variables other than $p$, these variables are assumed not to change during the discontinuous behavior. This, of course, should be verified in some way by the modeler of the system. Moreover, the variable $q$ is considered an auxiliary variable of an inductance, and is therefore also assumed not to change $q' = 0$.

The following constitutive processes describe an inductance, and dually a capacitor:

$$\mathrm{I}(\mathrm{I}) \approx \left[ \begin{array}{l} E' = \int_{p^-}^{p^+} \mathrm{I}(x)dx \\ q' = 0 \end{array} \right] \gg \Big( f = \mathrm{I}(p) \Big) \blacktriangleright \mathrm{I}(\mathrm{I})$$

$$\mathrm{C}(\mathrm{C}) \approx \left[ \begin{array}{l} E' = \int_{q^-}^{q^+} \mathrm{C}(x)dx \\ p' = 0 \end{array} \right] \gg \Big( e = \mathrm{C}(q) \Big) \blacktriangleright \mathrm{C}(\mathrm{C})$$

**Sources** A flow-source, is an element with a continuous constitutive relation $f = \mathrm{S}_f$, such that the function $\mathrm{S}_f$ satisfies $\frac{\partial \mathrm{S}_f}{\partial p} = 0$. A flow-source enforces

a certain flow, and has an arbitrary amount of energy, stored as generalized momentum, at its disposition to achieve this. Dually, an effort-source, is an element with a constitutive relation $e = \mathrm{S}_e$, such that the function $\mathrm{S}_e$ satisfies $\frac{\partial \mathrm{S}_e}{\partial q} = 0$.

For a flow-source, a similar reasoning as with the inductance, leads to $E' = \int_{p^-}^{p^+} \mathrm{S}_f \; dp = \mathrm{S}_f \cdot (p^+ - p^-) = \mathrm{S}_f \cdot p'$. As with inductances, it is assumed that if $\mathrm{S}_f$ depends on variables other than $p$, then these variables do not change. Also, $q$ is considered auxiliary, hence $q' = 0$. For an *effort-source*, we find that $E' = \int_{q^-}^{q^+} \mathrm{S}_e \; dq = \mathrm{S}_e \cdot q'$ and $p' = 0$. This leads to the following constitutive processes.

$$\mathrm{F}(\mathrm{S}_f) \approx \left[ \begin{array}{l} E' = \mathrm{S}_f \cdot p' \\ q' = 0 \end{array} \right] \gg \Big( \; f = \mathrm{S}_f \; \Big) \;\blacktriangleright\; \mathrm{F}(\mathrm{S}_f),$$

$$\mathrm{E}(\mathrm{S}_e) \approx \left[ \begin{array}{l} E' = \mathrm{S}_e \cdot q' \\ p' = 0 \end{array} \right] \gg \Big( \; e = \mathrm{S}_e \; \Big) \;\blacktriangleright\; \mathrm{E}(\mathrm{S}_e).$$

**Resistance** A resistance, is an element with a continuous constitutive relation $e = \mathrm{R}(f)$ such that the function R satisfies $x \cdot \mathrm{R}(x) \geq 0$ for all $x$. Combining this with the description of a bond, the equation $\dot{E} = e \cdot f = e \cdot \mathrm{R}(e) \geq 0$ is obtained, which illustrates the physical concept that a resistance dissipates energy.

For a *resistance*, both $p$ and $q$ are auxiliary. Furthermore, it is not possible to determine the dissipated amount of energy during the discontinuous behavior, because the amount of time from which is abstracted is unknown. In this paper, it is therefore assumed that resistances do not take part in the discontinuous behavior $E' = q' = p' = 0$. The models that are worked out in [Cuijpers et al. 2004], indicate that this is indeed a reasonable choice. For a resistance, the following constitutive process is obtained:

$$\mathrm{R}(\mathrm{R}) \approx \left[ \begin{array}{l} E' = 0 \\ p' = 0 \\ q' = 0 \end{array} \right] \gg \Big( \; e = \mathrm{R}(f) \; \Big) \;\blacktriangleright\; \mathrm{R}(\mathrm{R}).$$

**Junctions** The 0-junction represents conservation of total energy and total generalized displacement, at an equal change in generalized momentum. Dually, the 1-junction represents conservation of total energy and total generalized momentum at an equal change in generalized displacement. The discontinuous behavior is such that total generalized momentum and total generalized displacement are preserved or changing equally, respectively, and that also the total energy in the system is preserved.

Bear in mind, that the direction of the bond-arrows, determines the sign of the contribution of variables to the summations, as usual. To emphasize this, the notation $\pm$ is used, with every variable that should be positive when the power direction points inward and negative when outward. The following constitutive processes, denote junctions of a set $C$ of bonds/elements.

$$0 \approx \left[ \begin{array}{l} \sum_{c \in C} \pm E'_c = 0 \\ \sum_{c \in C} \pm q'_c = 0 \\ \forall_{c,c' \in C} p'_c = p'_{c'} \end{array} \right] \gg \left( \begin{array}{l} \sum_{c \in C} \pm f_c = 0 \\ \forall_{c,c' \in C} e_c = e_{c'} \end{array} \right) \;\blacktriangleright\; 0$$

$$1 \approx \left[ \begin{array}{l} \sum_{c \in C} \pm E'_c = 0 \\ \sum_{c \in C} \pm p'_c = 0 \\ \forall_{c,c' \in C} q'_c = q'_{c'} \end{array} \right] \gg \left( \begin{array}{l} \sum_{c \in C} \pm e_c = 0 \\ \forall_{c,c' \in C} f_c = f_{c'} \end{array} \right) \;\blacktriangleright\; 1$$

**Controlled junctions** In [Breedveld 1996, Mosterman et al.1998], controlled junctions were proposed to model a change in connection structure between elements. When active, a controlled junction acts like the junction it is associated with, but when inactive, it acts like a collection of 0-effort sources or a collection of 0-flow sources, depending on the type of the controlled junction.

In this paper, the notations 0/E, 1/E, 0/F and 1/F are used, to denote a 0-junction/0-effort element, a 1-junction/0-effort element, a 0-junction/0-flow and a 1-junction/0-flow element, respectively. Act and Inact are predicates over $\mathcal{V}_\mathrm{m}$ only, and model when a controlled junction is active or inactive, respectively. $\mathrm{Act}^-$ denotes the predicate Act where all variables $x \in \mathcal{V}_\mathrm{m}$ are replaced by $x^-$, and similarly for $\mathrm{Inact}^-$.

Abstracting small time-scale behavior is captured by a change in connection structure, which typically causes dissipation of energy (e.g., in a model of an inelastic collision). The description of controlled junctions models this dissipation as part of the constitutive process. When switching between Act and Inact, a decrease in the total energy of the system

is allowed. The discontinuous behavior of the other variables during this decrease, is governed by either the equations for the discontinuous behavior of the associated junction or the equations for the 0-effort or flow-source. When switching from one mode to another, both are possible.

The following hybrid constitutive process is obtained for the (1/E)-junction. The different modes are separated using the $\oplus$ operator. Four different flow-clauses describe the four continuous modes, and four different re-initialization clauses describe four discontinuous modes. However, the switching predicates only allow six possible combinations.

$$
\begin{array}{l}
\text{(1/E)(Act,Inact)} \approx \\
\left[\begin{array}{l} \text{Act}^- \\ \sum_{c\in C} \pm E'_c = 0 \\ \sum_{c\in C} \pm p'_c = 0 \\ \forall_{c,c'\in C} q'_c = q'_{c'} \end{array}\right] \gg \left(\begin{array}{l} \text{Act} \\ \sum_{c\in C} \pm e_c = 0 \\ \forall_{c,c'\in C} f_c = f_{c'} \end{array}\right) \blacktriangleright \text{(1/E)} \\
\oplus \\
\left[\begin{array}{l} \text{Inact}^- \\ \forall_{c\in C} \left\{\begin{array}{l} E'_c = 0 \\ p'_c = 0 \end{array}\right. \end{array}\right] \gg \left(\begin{array}{l} \text{Inact} \\ \forall_{c\in C} e_c = 0 \end{array}\right) \blacktriangleright \text{(1/E)} \\
\oplus \\
\left[\begin{array}{l} \text{Inact}^- \\ \sum_{c\in C} \pm E'_c \geq 0 \\ \sum_{c\in C} \pm p'_c = 0 \\ \forall_{c,c'\in C} q'_c = q'_{c'} \end{array}\right] \gg \left(\begin{array}{l} \text{Act} \\ \sum_{c\in C} \pm e_c = 0 \\ \forall_{c,c'\in C} f_c = f_{c'} \end{array}\right) \blacktriangleright \text{(1/E)} \\
\oplus \\
\left[\begin{array}{l} \text{Act}^- \\ \sum_{c\in C} \pm E'_c \geq 0 \\ \sum_{c\in C} \pm p'_c = 0 \\ \forall_{c,c'\in C} q'_c = q'_{c'} \end{array}\right] \gg \left(\begin{array}{l} \text{Inact} \\ \forall_{c\in C} e_c = 0 \end{array}\right) \blacktriangleright \text{(1/E)} \\
\oplus \\
\left[\begin{array}{l} \text{Inact}^- \\ \forall_{c\in C} \left\{\begin{array}{l} \pm E'_c \geq 0 \\ p'_c = 0 \end{array}\right. \end{array}\right] \gg \left(\begin{array}{l} \text{Act} \\ \sum_{c\in C} \pm e_c = 0 \\ \forall_{c,c'\in C} f_c = f_{c'} \end{array}\right) \blacktriangleright \text{(1/E)} \\
\oplus \\
\left[\begin{array}{l} \text{Act}^- \\ \forall_{c\in C} \left\{\begin{array}{l} \pm E'_c \geq 0 \\ p'_c = 0 \end{array}\right. \end{array}\right] \gg \left(\begin{array}{l} \text{Inact} \\ \forall_{c\in C} e_c = 0 \end{array}\right) \blacktriangleright \text{(1/E)}
\end{array}
$$

The definitions for 0/E, 0/F and 1/F are dual to this.

Note that in [Stromberg et al. 1993], a different approach to hybrid modeling using 'switch elements' was proposed. In the context of this work, these elements are expressible in terms of controlled junctions [Cuijpers et al. 2004].

# 4 Example

To illustrate, a collision between two masses is modeled, the bond graph of which is depicted in Fig. 1. In [Cuijpers et al. 2004], more complex examples are discussed.

For the masses, linear inductances $\text{I}(p) = \frac{p}{m}$ are used. The controlled 1/E-junction models the exchange of momentum between the two masses during collision. It is only active when the position $q$ of the masses is equal, and either the velocity $f$ of the left mass is greater than that of the right mass, or the velocities are equal and the acceleration $\dot{f} = \frac{e}{m}$ of the left mass is greater than that of the right mass (assuming that the positive direction of $e$ and $f$ is from left to right).

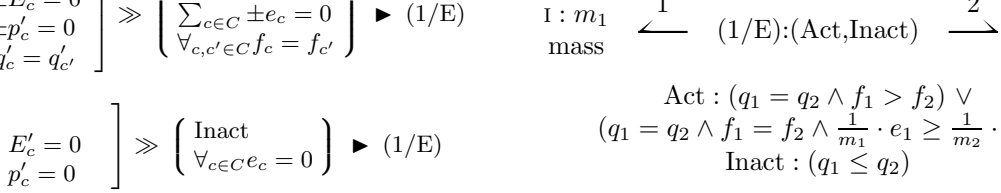


Figure 1: Bond graph for a collision

The system is modeled by the following parallel composition of constitutive processes. The subscripts indicate the numbers of the bonds with which the processes are associated. The minus signs indicate those bonds for which variables should be negated.

$$
\begin{array}{l}
\text{Col} \approx \\
\text{I}_1(\frac{\cdot}{m_1}) \parallel \text{B}_1 \parallel \text{(1/E)}_{\{-1,-2\}}\text{(Act,Inact)} \parallel \text{B}_2 \parallel \text{I}_2(\frac{\cdot}{m_2})
\end{array}
$$

Symbolic manipulation of the above description (see [Cuijpers and Reniers]) leads to elimination of the parallel composition. If bond definitions are left implicit we get:

$$
\begin{array}{l}
\text{Col} \approx \\
\left[\begin{array}{l} E'_1 = \frac{(p_1^+)^2 - (p_1^-)^2}{2 \cdot m_1} \\ E'_2 = \frac{(p_2^+)^2 - (p_2^-)^2}{2 \cdot m_2} \\ E'_1 + E'_2 = 0 \\ p'_1 + p'_2 = 0 \\ q'_1 = q'_2 = 0 \\ \text{Act}^- \end{array}\right] \gg \left(\begin{array}{l} p_1 = m_1 \cdot f_1 \\ p_2 = m_2 \cdot f_2 \\ e_1 + e_2 = 0 \\ f_1 = f_2 \\ \text{Act} \end{array}\right) \blacktriangleright \text{Col} \\
\oplus
\end{array}
$$

$$\left[\begin{array}{l} E_1' = E_2' = 0 \\ p_1' = p_2' = 0 \\ q_1' = q_2' = 0 \\ \text{Inact}^- \end{array}\right] \gg \left(\begin{array}{l} p_1 = m_1 \cdot f_1 \\ p_2 = m_2 \cdot f_2 \\ e_1 = e_2 = 0 \\ \text{Inact} \end{array}\right) \blacktriangleright \text{Col}$$

$$\oplus$$

$$\left[\begin{array}{l} E_1' = \frac{(p_1^+)^2 - (p_1^-)^2}{2 \cdot m_1} \\ E_2' = \frac{(p_2^+)^2 - (p_2^-)^2}{2 \cdot m_2} \\ E_1' + E_2' \leq 0 \\ p_1' + p_2' = 0 \\ q_1' = q_2' = 0 \\ \text{Inact}^- \end{array}\right] \gg \left(\begin{array}{l} p_1 = m_1 \cdot f_1 \\ p_2 = m_2 \cdot f_2 \\ e_1 + e_2 = 0 \\ f_1 = f_2 \\ \text{Act} \end{array}\right) \blacktriangleright \text{Col}$$

$$\oplus$$

$$\left[\begin{array}{l} E_1' = \frac{(p_1^+)^2 - (p_1^-)^2}{2 \cdot m_1} \\ E_2' = \frac{(p_2^+)^2 - (p_2^-)^2}{2 \cdot m_2} \\ E_1' + E_2' \leq 0 \\ p_1' + p_2' = 0 \\ q_1' = q_2' = 0 \\ \text{Act}^- \end{array}\right] \gg \left(\begin{array}{l} p_1 = m_1 \cdot f_1 \\ p_2 = m_2 \cdot f_2 \\ e_1 = e_2 = 0 \\ \text{Inact} \end{array}\right) \blacktriangleright \text{Col}$$

The resulting description is in the form of a constitutive process again, with four instead of six modes because some re-initialization behavior coincided. Note that the model describes the entire spectrum between perfectly elastic and inelastic collisions, because an arbitrary energy dissipation is allowed in the controlled junctions. This reflects the idea that the modeler abstracted from the precise parameters that govern the impact. If necessary, our theory can be adapted to model specific kinds of collision, but this is beyond the scope of this paper.

# 5 Conclusions

In this paper, a process algebraic foundation for hybrid bond graphs is developed. It shows that discontinuous constitutive relations can often be derived from continuous constitutive relations through substitution and integration, and that a change in connection structure, may lead to a dissipation of energy. The exact energy loss is not specified, which sometimes results in a model with branching behavior.

The added value of the use of process algebra for the description of hybrid systems is, amongst others, the possibility to perform symbolic reasoning. For continuous systems, it is useful to construct constitutive equations, and then, through symbolic reasoning, rewrite those equations into a form (known as *ordinary differential equations*) that enables the modeler to perform simulations or other kinds of analysis. Using HyPA, these techniques can be extended to hybrid systems as well. Currently, research is going on to automate the rewriting of HyPA processes into an appropriate normal form, based on earlier work for the process algebra $\mu$CRL [Usenko 2002]. Ultimately, such rewriting algorithms could also be implemented into tools for simulation languages like hybrid $\chi$, which have a process algebraic foundation [Schiffelers et al. 2003].

One of the added values of bond graphs, is that it facilitates the symbolic reasoning with a graphical interface. An example, is that there is a set of bond graph reduction rules, that lead to equivalent bond graphs modulo elimination of variables associated with connections between junctions. Based on what is known so far about variable elimination in HyPA, we found that the existing graph reduction rules are still valid for the hybrid bond graphs. However, the formal details still need to be worked out. It is expected that process algebraic reasoning can also be used to develop new graph reduction rules that deal specifically with hybrid elements.

Another example, is the theory of causality for standard bond graphs, that aids in the construction of ordinary differential equations, by assigning an input/output status to model variables. Unfortunately, this theory depends on the connection structure between elements, which makes it hard to extend it to the hybrid case. In HyPA, it is possible to treat different connection structures separately. Standard causality theory can be used for each of the continuous modes, but new theory needs to be developed for the discontinuities that occur in between. It is, expected that the constitutive process foundation that is given in this paper, will be helpful in the development of causality theory for hybrid bond graphs.

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