

# Sliding Mode Control and Feedback Linearization for Non-regular Systems

Fu Zhang\* Benito R. Fernández\*\* Pieter J. Mosterman\*  
Timothy Josserand\*\*

\* *The MathWorks, Inc. Natick, MA, 01760*

\*\* *University of Texas at Austin, Austin, TX, 78712*

---

**Abstract:** This paper presents an approach for sliding mode control(SMC) and feedback linearization(FBL) of systems with relative order singularities. Traditionally, SMC and FBL are designed by taking derivatives of the output until the control signal appears(at the  $r_1^{\text{th}}$  derivative). When a system does not have a well-defined relative degree ( $r_1$ ), the coefficient that multiplies  $u$  vanishes for some region of the state-space  $\mathcal{S}_1$ . In this instance, conventional SMC and FBL techniques fail. The presented approach differentiates further the output until the control input appears again (the secondary relative degree,  $r_2$ ) and a differential equation in  $u$  is acquired. It may be possible to solve for a dynamic compensator, or in the neighborhood of the singularity,  $\mathcal{N}_1$ , the equations degenerate to a polynomial form. Preliminary results show that at the singularity region,  $\mathcal{S}_1$ , the control-derivative term disappears and the differential equation is degenerated to a center manifold defined by a polynomial (quadratic in general) equation on  $u$ . The solution to the quadratic equations is discussed. When this equation has only real roots, the system is well defined at the singularity. A switching controller can be designed to switch from the  $r_1^{\text{th}}$  controller when system is far away from the singularity to the  $r_2^{\text{th}}$  controller when the system is in the neighborhood of the singularity. We demonstrate the controller applied to the ball and beam system.

---

## 1. INTRODUCTION

The ball and beam (B&B) system (Fig. 1) is one of the most popular models for studying control systems because of its simplicity and yet the control techniques that can be studied cover many important control methods (Barbu et al. [1997], Hauser et al. [1992], Lai et al. [1994], Leith and Leithead [2001], Marra et al. [1996], Tomlin and Sastry [1997], Yi et al. [1996]).

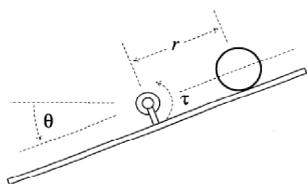


Fig. 1. The ball beam system

The (B&B) system is non-regular, i.e., the relative degree of it is not well defined at certain locations in the phase space. This interesting property, common to other difficult-to-solve control systems, has motivated much research in the past. Thus, conventional exact feedback linearization(eFBL) and sliding mode control(SMC), are hard or simply do not apply. The well-known work of Hauser et. al. (Hauser et al. [1992]), used approximation feedback linearization(aFBL) by disregarding certain terms that lead to the singularity. However, this approach does not

work well when the system is away from the singularity, because of the approximation error that is generated by disregarding the terms. Tomlin et. al. (Tomlin and Sastry [1997]) proposed a switching control law: a controller that uses eFBL when the system is in the region far away from the singularity and a switch to the aFBL controller when the system approaches the singularity. Lai et. al. (Lai et al. [1994]), proposed a tracking controller based on approximate backstepping (aBS) that has better steady state error than other approximation methods. Other approaches for the B&B control problem include a fuzzy controller (Yi et al. [1996]) and a genetic controller (Marra et al. [1996]).

This paper tries to generalize the control problem to non-regular systems. The main idea is to define multiple relative orders  $r_k$  and propose controllers (FBL or SMC) for each of the relative orders and create a control law that switches when the system approaches the singularity neighborhood  $\mathcal{S}_k$  associated with each relative order. The result is a group of possible switching controllers. The specific desired controller structure will depend on the particular system characteristics.

The technique is applied to the B&B problem, generating a switching control law similar to the switching controller presented in (Tomlin and Sastry [1997]). The contribution of this paper is that by taking  $(r+1)^{\text{th}}$  ( $r$  is the relative degree of the system away from singularities) derivatives, it shows that the neighborhood of the singularity can be further divided into two regions. In one of the region, the exact system still have a well-defined relative degree. While approaches in (Hauser et al. [1992]) and (Tomlin and Sastry [1997]), the exact system is not well-defined in the

whole region. Our controller, by dividing the neighborhood into two regions, is able to realize a more precise control.

This paper is organized as follows: Section 2 is a brief description of the supporting methods presented in (Hauser et al. [1992], Tomlin and Sastry [1997]). Section 3 presents the high order derivative approach, with the ball and beam system used as an example. Section 4 shows how to extend the high order method to SMC. Section 5 is the discussion of the presented approach and the behavior of a non-regular system, Section 6 gives some simulation results and Section 7 is the generalized formulation of the presented method to solve feedback linearization with singularities. Section 8 presents the summary and some open questions.

## 2. APPROXIMATION FEEDBACK LINEARIZATION

Figure 1 shows the ball and beam schematic as a specific sample of the class of systems under investigation (namely, non-regular systems). The controller input  $\tau$  rotates the beam with the ball on it, around a pivot point. The ball rolls based on the gravitational pull projected by the beam's angle,  $\theta$ . The objective of the controller is to maintain the ball at a distance  $r_d$  from the pivot point.

We assume that the ball is always in contact with the beam. Without loss of generality, we can neglect the contact friction. Using a nonlinear transformation, the equations of the system can be written as (Hauser et al. [1992]):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ B(x_1x_4^2 - g \sin x_3) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot u$$

$$y = x_1 = h(\mathbf{x}) \quad (1)$$

where,  $x_1(t) = r(t)$  is the ball position measured from the beam center to the ball center,  $x_2(t) = \dot{r}(t) = v(t)$  is the ball relative velocity to the beam,  $x_3(t) = \theta(t)$  is the beam angle, and  $x_4(t) = \dot{\theta}(t) = \omega(t)$  is the beam's angular velocity.  $B$  and  $g$  are constants and the input  $u$  is a nonlinear transformation of the torque  $\tau$ .

Using the typical steps for feedback linearization (FBL) or sliding mode control (SMC), take the derivatives of  $y$  till  $u$  appears at the right-hand side, we obtain:

$$\begin{aligned} y &= \mathcal{L}_{\mathbf{f}}^0(h) = x_1 = h(\mathbf{x}) \\ \dot{y} &= \mathcal{L}_{\mathbf{f}}^1(h) = x_2 = L_{\mathbf{f}}h(\mathbf{x}) \\ \ddot{y} &= \mathcal{L}_{\mathbf{f}}^2(h) = B(x_1x_4^2 - g \sin(x_3)) \\ \dddot{y} &= \underbrace{B(x_2x_4^2 - gx_4 \cos x_3)}_{\mathcal{L}_{\mathbf{f}}^3(h)} + \underbrace{2Bx_1x_4}_{\mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^2(h))} u \end{aligned} \quad (2)$$

where  $\mathcal{L}_{\mathbf{f}}^k(h)$  is the  $k^{\text{th}}$  Lie derivative of  $h$  along  $\mathbf{f}$  (Fernández-Rodríguez [1998]). Here,  $u$  appears at the right-hand side of the 3<sup>rd</sup> derivative of  $y$ , hence the relative degree of the system is 3, except at the singularity  $\mathcal{S}_1 = \{\forall \mathbf{x} \in \mathcal{R}^n \mid \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^2(h(\mathbf{x}))) = 0\}$ . At this point,  $u$  disappears from the right-hand side of the derivative, and the relative degree is not well defined.

If we want to FBL, we can make  $\ddot{y} = v$ , then

$$u = \frac{v - B(x_2x_4^2 - gx_4 \cos(x_3))}{2Bx_1x_4} = \frac{v(t) - \alpha(\mathbf{x})}{\beta(\mathbf{x})} \quad (3)$$

The aFBL technique presented by Hauser (Hauser et al. [1992]) proposes two approximation methods. The first one is to disregard the  $x_1x_4^2$  term in Eq. (2) and then differentiate the output  $y$  until  $u$  appears (again):

$$\begin{aligned} \xi_1 &= y = x_1 \\ \dot{\xi}_1 &= \xi_2 = \dot{y} = x_2 \\ \dot{\xi}_2 &= \xi_3 = \ddot{y} = \underbrace{Bx_1x_4^2}_{\text{Disregards this term}} - Bg \sin(x_3) \\ \dot{\xi}_3 &= \xi_4 \approx \dddot{y} = -Bgx_4 \cos(x_3) \\ \dot{\xi}_4 &= y^{[4]} = Bgx_4^2 \sin(x_3) - Bg \cos(x_3)u \\ u &= \frac{v - Bgx_4^2 \sin(x_3)}{-Bg \cos(x_3)} = \frac{v(t) - \alpha(\mathbf{x})}{\beta(\mathbf{x})} \end{aligned} \quad (4)$$

The other approximation is to disregard the  $2Bx_1x_4u$  term in Eq. (2) and then take the 4<sup>th</sup> derivative of  $y$ :

$$\begin{aligned} y &= x_1 \\ \dot{y} &= x_2 \\ \ddot{y} &= B(x_1x_4^2 - g \sin(x_3)) \\ \dddot{y} &= B(x_2x_4^2 - gx_4 \cos(x_3)) + \underbrace{2Bx_1x_4}_{\text{Disregards this term}} u \\ y^{[4]} &= B^2x_1x_4^4 + B(1-B)x_4^2 \sin(x_3) + (-Bg \cos(x_3) + 2Bx_2x_4)u \\ u &= \frac{v - (B^2x_1x_4^4 + B(1-B)gx_4^2 \sin(x_3))}{B(2x_2x_4 - g \cos(x_3))} = \frac{v(t) - \alpha(\mathbf{x})}{\beta(\mathbf{x})} \end{aligned} \quad (6)$$

This approximation approach (Hauser et al. [1992]) works well when system is far away from the  $\mathcal{S}_1$ .

In (Hauser et al. [1992]), both approximations disregarded the terms that lead to singularity before taking the 4<sup>th</sup> derivative of  $y$  and thus effectively adding modeling errors. These techniques are compared in a later section with our proposed approach and are further discussed then.

## 3. APPROXIMATION USING HIGH ORDER DERIVATIVES

An alternative is to take the 4<sup>th</sup> derivative of  $y$  without disregarding the nonlinear terms. This is typically not done because it results in a differential equation of  $u$  that could be difficult to solve and because it creates a dynamic compensator. Following the general feedback linearization procedures and differentiating the output one more time, we obtain  $\dot{u}$  at the right-hand side:

$$\begin{aligned} \xi_1 &= y = x_1 \\ \dot{\xi}_1 &= \xi_2 = \dot{y} = x_2 \\ \dot{\xi}_2 &= \xi_3 = \ddot{y} = B(x_1x_4^2 - g \sin(x_3)) \\ \dot{\xi}_3 &= \xi_4 = \dddot{y} = B(x_2x_4^2 - gx_4 \cos(x_3) + 2x_1x_4u) \\ \dot{\xi}_4 &= y^{[4]} = B[\dot{x}_2x_4^2 + 2x_2x_4\dot{x}_4 - g(\dot{x}_4 \cos(x_3) - x_4\dot{x}_3 \sin(x_3)) \\ &\quad + 2(\dot{x}_1x_4 + \dot{x}_4x_1)u + \underbrace{2\dot{u}x_1x_4}_{\dot{u}\text{-term}}] \end{aligned} \quad (7)$$

Substitute the state space equations (1) into (7), yields:

$$y^{[4]} = B[(Bx_1x_4^2 - Bg\sin(x_3))x_4^2 + 2x_2x_4u - g(u\cos(x_3) - x_4^2\sin(x_3)) + 2(x_2x_4 + ux_1)u + \underbrace{2\dot{u}x_1x_4}] \quad (8)$$

This term vanishes at singularity  $s_1$

Around the neighborhood of the singular point,  $\mathcal{N}_1$ , where,  $\mathcal{N}_1 = \{\forall \mathbf{x} \in \mathcal{R}^n, \mathbf{z} \in \mathcal{S}_1 \mid \|\mathbf{x} - \mathbf{z}\| < \delta^2\}$  (9) we have  $x_1x_4 \rightarrow 0$  and Eq. (8) becomes:

$$y^{[4]} = B [Bx_1x_4^4 - Bg\sin(x_3)x_4^2 + 2x_2x_4u - g(u\cos(x_3) - x_4^2\sin(x_3)) + 2(x_2x_4 + ux_1)u] \quad (10)$$

Let  $v = y^{[4]}$ , collecting the terms, a quadratic equation of  $u$  at the singular point results:

$$[2Bx_1]u^2 + [B(4x_2x_4 - g\cos(x_3))]u + [B^2x_1x_4^4 + Bg(1 - B)\sin(x_3)x_4^2 - v] = 0 \quad (11)$$

If the general FBL procedure is applied and the 4<sup>th</sup> order derivative of the output is calculated, this yields Eq. (8), a differential equation in  $u$ . This results in a dynamic controller that may be implemented outside  $\mathcal{N}_1$ . However, close to the singular point,  $\mathbf{x} \in \mathcal{N}_1$ ,  $x_1x_4 \rightarrow 0$ , and the differential equation degenerates to the quadratic equation Eq. (11) that can be used to solve for  $u$ . Further more, Eq. (11) can be used to approximate the system around the neighborhood of the singularity,  $\mathcal{N}_1$ .

By using the switching idea introduced in (Tomlin and Sastry [1997]), a switching controller can be designed using Eq. (3) when  $\mathbf{x} \in \mathcal{N}_1$  and Eq. (11) when  $\mathbf{x} \notin \mathcal{N}_1$ . Unlike (Hauser et al. [1992]) and (Tomlin and Sastry [1997]), Eq. (11) is an "exact" feedback linearization of the original system at the singularity without disregarding the terms that lead to the singularity.

Since Eq. (11) is a quadratic equation, the general solutions are two conjugate complex roots. Define:

$$\begin{aligned} a(t) &= 2Bx_1 \\ b(t) &= B(4x_2x_4 - g\cos(x_3)) \\ c(t) &= B^2x_1x_4^4 + Bg(1 - B)x_4^2\sin(x_3) - v \\ \Delta &= b^2 - 4ac \\ u &= \frac{-b \pm \sqrt{\Delta}}{2a} \end{aligned} \quad (12)$$

Solutions to Eq. (11) depend on the value of  $\Delta$  in Eq. (12). In order to implement Eq. (11),  $\Delta \geq 0$ . It is difficult to find the conditions that guarantee  $\Delta \geq 0$ . However, the following section shows that Eq. (11) has only real solutions in some neighborhood of the singularity. In the event of having complex conjugates, one may use the real part of the solution of Eq. (12), i.e.,

$$u = -B(4x_2x_4 - g\cos(x_3)) + v \quad (13)$$

which cancels the open-loop dynamics of the system.

#### 4. SMC THROUGH SINGULARITIES

For Sliding Mode Control, we use the same steps for feedback linearization (FBL), taking the derivatives of  $y$

until  $u$  appears at the right side, we obtain Eq. (2) where  $u$  appears at the right-hand side of the 3<sup>rd</sup> derivative of  $y$  hence the relative degree of the system is 3, except at the singularity  $\mathcal{S}_1$ . For SMC, we define a sliding surface as a stable differential operator of order  $r_1 - 1$  on  $y$ .

$$\mathcal{S}(t) = \left\{ \mathbf{x} \in \mathcal{R}^n \mid \mathbf{s}(t) \triangleq \mathbf{s}(\mathbf{x}) = 0 \right\} \quad (14)$$

where,  $\mathbf{s}(\mathbf{x})$  is referred to as the surface parameter vector. The order of the surface operator is one less than the relative order  $r_k$  of the system. This is done so the surface attractiveness can be guaranteed. The surface parameter vector  $\mathbf{s}$  is defined as follows

$$\mathbf{s}(\mathbf{x}) = \mathcal{D}^k(y) = \sum_{i=0}^{r_k-1} \lambda_i \frac{d^i}{dt^i} \tilde{y}(t) \quad (15)$$

where  $\mathcal{D}^k(\cdot)$  is a linear differential operator that defines the surface dynamics with coefficients  $\lambda_i$  (coefficients or the characteristic equation), and  $\tilde{y}(t) = y_d(t) - y(t)$  is the output tracking error ( $y_d(t)$  is the desired output trajectory).

Existence of the *sliding mode* requires the *attractiveness condition*:

$$\lim_{\mathbf{s}(t) \rightarrow 0} \mathbf{s}^T(t) \cdot \dot{\mathbf{s}}(t) < 0 \quad (16)$$

Differentiating the surface parameter vector  $\mathbf{s}(t)$  and forcing its time derivative to a function in the 2<sup>nd</sup> and 4<sup>th</sup> quadrant,

$$\dot{\mathbf{s}}(t) = -\eta \sigma\left(\frac{\mathbf{s}(t)}{\mu}\right) = \dot{\mathbf{s}}_d(t) \quad (17)$$

the surface satisfies the existence condition in Eq. (16). The parameter  $\mu$  is introduced to adjust the "boundary layer" of the surface. The squashing function,  $\sigma(\cdot)$  proposed is the hyperbolic tangent, defined by

$$\sigma(\zeta) = \frac{1 - e^{-\zeta}}{1 + e^{-\zeta}} \quad (18)$$

$$\begin{aligned} \dot{\mathbf{s}}_d(t) &= \frac{d}{dt} \mathcal{D}^k(y) = \sum_{i=0}^{r_k-1} \lambda_i \frac{d^{i+1}}{dt^{i+1}} \tilde{y}(t) = -\eta \sigma\left(\frac{\mathbf{s}(t)}{\mu}\right) \\ &= \sum_{i=0}^{r_k-1} \lambda_i y_d^{[i+1]}(t) - \sum_{i=0}^{r_k-2} \lambda_i y^{[i+1]}(t) + \lambda_{r_k-1} y^{[r_k]}(t) \\ &= \sum_{i=0}^{r_k-1} \lambda_i y_d^{[i+1]}(t) - \sum_{i=0}^{r_k-1} \lambda_i \mathcal{L}_{\mathbf{f}}^{i+1}(h) - \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_k-1}(h))u \\ &= \mathbf{\Gamma}(y_d) + \boldsymbol{\alpha}(\mathbf{x}) + \boldsymbol{\beta}(\mathbf{x})u \end{aligned} \quad (19)$$

next, we solve for  $u(t)$  from Eq. (19)

$$\begin{aligned} u(t) &= \frac{\dot{\mathbf{s}}_d(t) - \mathbf{\Gamma}(y_d) - \boldsymbol{\alpha}(\mathbf{x})}{\boldsymbol{\beta}(\mathbf{x})} \\ &= \frac{v(t) - \boldsymbol{\alpha}(\mathbf{x})}{\boldsymbol{\beta}(\mathbf{x})} \end{aligned} \quad (20)$$

where,

$$v(t) = -\eta \sigma\left(\frac{\mathbf{s}(t)}{\mu}\right) - \sum_{i=0}^{r_k-1} \lambda_i y_d^{[i+1]}(t) \quad (21)$$

Hence, all results from FBL, including using higher derivatives mentioned in above section, can be extended to SMC by replacing  $v(t)$  in the FBL framework by the  $v(t)$  of Eq. (21).

## 5. BEHAVIOR AROUND THE SINGULARITY

The singularity and its neighborhood  $\mathcal{N}_1$  are shown in Fig. 2. In the  $x_1$ - $x_4$  phase plane it can be divided into two regions:

$$\begin{aligned} \mathcal{S}_a: & \quad x_1 = 0, \quad x_2 \in \mathcal{R} \quad \text{In Fig. 5, it is the } x_4\text{-axis.} \\ \mathcal{S}_b: & \quad x_4 = 0, \quad x_4 \in \mathcal{R} \quad \text{In Fig. 5, it is the } x_1\text{-axis.} \end{aligned}$$

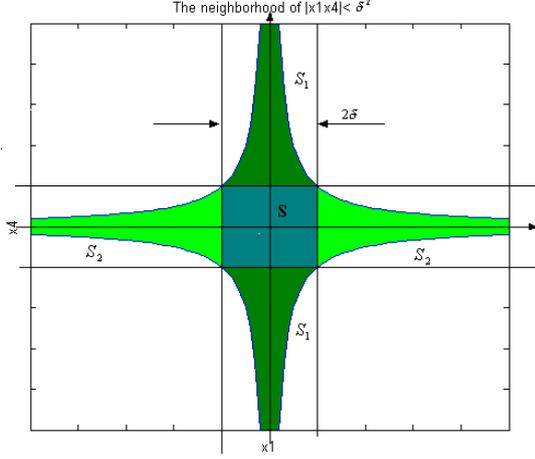


Fig. 2. The singularity region  $\mathcal{S}_1$  are both axes. The neighborhood  $\mathcal{N}_1$  of the singularity region is enclosed by the hyperbolas  $x_1x_4 = \pm\delta^2$ .

In the  $x_1$ - $x_4$  phase plane,  $\mathcal{N}_1$  is the shadowed region surrounded by four hyperbolic curves:  $|x_1x_4| = \delta^2$ . The neighborhood of  $\mathcal{S}_1$  can also be divided into two regions:

$$\begin{aligned} \mathcal{S}_\alpha &= \{x \in \mathcal{N}_1 \mid \|x_1\| \leq \delta\} \subset \mathcal{N}_1 \\ \mathcal{S}_\beta &= \mathcal{N}_1 - \mathcal{S}_\alpha \subset \mathcal{N}_1 \end{aligned}$$

**Condition (1)** when system falls in  $|x_1| \leq \delta$ , Eq. (11) can be approximated by

$$\underbrace{2Bx_1u^2}_{\text{disregarded, when } x_1 \rightarrow 0} + B(4x_2x_4 - g \cos(x_3)) + [B^2x_1x_4^4 + Bg(1-B) \sin(x_3)x_4^2 - v] = 0 \quad (22)$$

Thus,

$$u_{\mathcal{S}_\alpha} = \frac{v - [B^2x_1x_4^4 + B(1-B)gx_4^2 \sin(x_3)]}{B(4x_2x_4 - g \cos(x_3))} \quad (23)$$

Comparing with the aFBL expression of  $u$  (Eq. 6) used in (Hauser et al. [1992]), the only difference of the two equations is the factor of the  $Bx_2x_4$  term. In Eq. (23), the factor is 4 and in Eq. (6) is 2. This difference is because of the disregarded term  $2Bx_1x_4u$  in aFBL.

More importantly, it shows that Eq. (6) only captures the system when  $x_1 \rightarrow 0$  while it tries to approximate the system when  $x_1x_4 \rightarrow 0$ . In other words, approximation by disregarding the nonlinear term before taking  $(r+1)$ <sup>th</sup> derivatives (Hauser et al. [1992], Tomlin and Sastry [1997]) is only a partial approximation of system near the singularity.

**Condition (2).** When system falls into  $|x_4| < \delta$ . Eq. (11) can be approximated by

$$2Bx_1u^2 + uB(-g \cos(x_3)) - v = 0 \quad (24)$$

In this case, the solutions to Eq. (10) is a pair of conjugate complex roots. The condition for the above equation to have only real roots is:

$$\Delta = (Bg \cos(x_3))^2 + 8Bvx_1 \geq 0 \quad (25)$$

The above condition Eq. (25) is a paraboloid (in the  $x_1$ - $x_3$  subspace) bifurcation surface. Outside the paraboloid sub-region, Eq. (24) will only have real roots and inside only complex-conjugate roots. Further study in this region is needed. Currently, as a heuristic rule, we suggest to use the real part of  $u$ . Thus, the neighborhood of the singularity can be approximated by Eq. (11) and Eq. (24).

The switching of controllers can be designed so that: When  $x \notin \mathcal{N}_1$ , exact linearization Eq. (3) is used. when  $x \in \mathcal{S}_\alpha$ , Eq. (23), which is also exact linearization, will be used. When  $x \in \mathcal{S}_\beta$ , Eq. (24) is used. Previous work (Tomlin and Sastry [1997]) provides the applicability of such switching law based on the zero dynamics at the switching boundary.

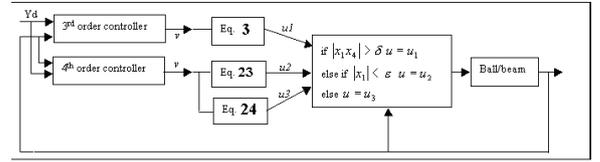


Fig. 3. The switching controller

Figure 3 shows the switching controller using the aforementioned scheme.

## 6. SIMULATION RESULTS

Several test cases used in (Hauser et al. [1992], Lai et al. [1994], Tomlin and Sastry [1997], Yi et al. [1996]) are also used for comparison purpose. Simulink<sup>®</sup> (Mat [2004]) was used for simulations with the 3<sup>rd</sup> order and 4<sup>th</sup> order controllers that are designed with all the poles at  $-2$ .  $B=0.7143$  and  $g=9.8$ . For comparison purpose, the following four cases are simulated:

- ① Regulation of the system to the equilibrium point  $[0 \ 0 \ 0 \ 0]$ . The same examples are used in (Yi et al. [1996]).
  - (1a) Initial conditions close to singularity: The switching condition is  $\delta^2 = |x_1x_4| = 0.02$ . The simulation results are shown in Fig. 4. The Results are similar to (Hauser et al. [1992], Yi et al. [1996]).
  - (1b) Initial conditions away from the singularity: The switching condition is  $\delta^2 = |x_1x_4| = 1$ . The simulations are shown in Fig. 6. As pointed out in (Yi et al. [1996]), the approximation method used in (Hauser et al. [1992]) failed on set(1b). Compared to the results in (Yi et al. [1996]), (compare Fig. 6 in (Yi et al. [1996]) and Fig. 5 in this paper), the presented method gives faster regulation.
- This test case shows that the presented method can regulate the system, even from initial conditions away from the singularity, with faster response.
- ② Tracking periodic functions, the switching condition is  $\delta^2 = |x_1x_4| = 0.02$ .

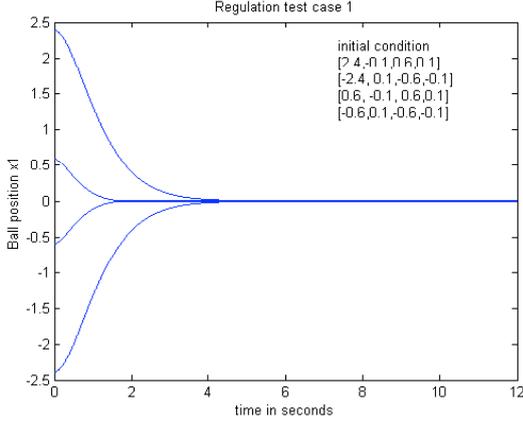


Fig. 4. Results of set (1a), regulation with  $\mathbf{x}_o$  close to  $\mathcal{N}_1$ .

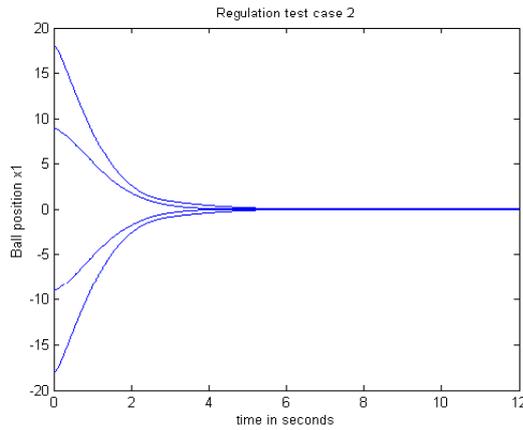


Fig. 5. Set (1b) regulation to origin results.  $\mathbf{x}_o$  far from  $\mathcal{N}_1$ . Similar simulation with method from (Hauser et al. [1992]) fails for these cases.

(2a)  $y_d = 1.9 \sin(1.3t) + 3$ ,  $\mathbf{x}_o = [3, 0, 0, 0]$ . This case is used in (Tomlin and Sastry [1997]) and the approximation method presented in (Hauser et al. [1992]) is unstable. Fig. 6 shows the tracking results, with  $y_d = 1.9 \sin(1.3t) + 3$ ,  $\mathbf{x}_o = [3, 0, 0, 0]$ . The transient period is short and then the system quickly tracks the target with a maximum steady state error equals to  $5e-5$ .

Although the method in (Tomlin and Sastry [1997]) is stable, the steady state error is quite large. (Compare Fig. 3 in (Tomlin and Sastry [1997]) and Fig. 6 in this paper).

(2b)  $y_d = 3 \cos(\frac{\pi}{5}t)$ ,  $\mathbf{x} = [3, 0, 0, 0]$ , which is used in (Hauser et al. [1992], Lai et al. [1994]), Fig. 7 shows the tracking results for this case. The maximum steady state error is  $1.5e-3$ .

The results above demonstrate the benefits of the proposed algorithm as compared to the reviewed literature.

## 7. GENERAL FORMULATION OF FEEDBACK LINEARIZATION WITH SINGULARITY

Here, we try to generalize the formulation to a class of SISO nonlinear systems of the form:

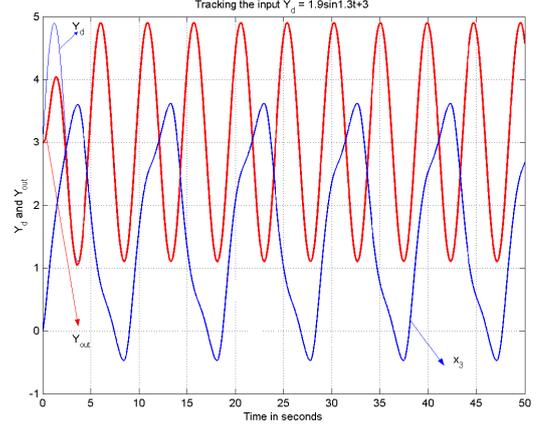


Fig. 6. Tracking of  $y_d = 1.9 \sin(1.3t) + 3$ ,  $\mathbf{x}_o = [3, 0, 0, 0]$

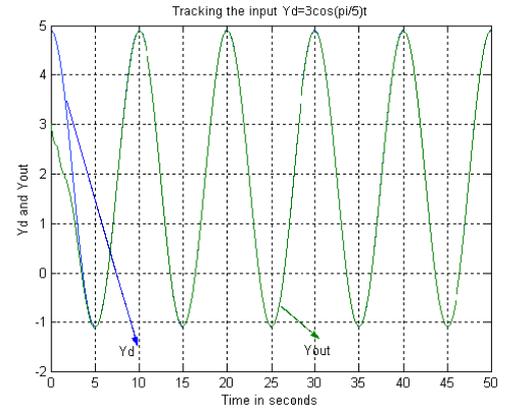


Fig. 7. Tracking of  $y_d = 3 \cos(\frac{\pi}{5}t)$ ,  $\mathbf{x} = [3, 0, 0, 0]$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot u$$

$$y(t) = h(\mathbf{x}) \quad (26)$$

where,  $\mathbf{x} \in \mathcal{R}^n$ ,  $u \in \mathcal{R}^1$ ,  $y \in \mathcal{R}^1$ , are the states, input, and output of the system. If the system has relative order of the output  $r_1$ , then,

$$\mathcal{L}_g(\mathcal{L}_f^k(h)) = 0 \quad 1 < k < r_1 - 2$$

$$\mathcal{L}_g(\mathcal{L}_f^{r_1-1}(h)) \neq 0 \quad \forall \mathbf{x} \notin \mathcal{S}_1 \subset \mathcal{R}^n$$

For exact I/O linearization,  $\exists r_1 \leq n$ . Then, following the general procedure, we obtain:

$$\xi_1 = h(\mathbf{x}) = y = \mathcal{L}_f^0(h)$$

$$\xi_2 = \dot{\xi}_1 = \dot{y} = \mathcal{L}_f^1(h)$$

$$\xi_3 = \dot{\xi}_2 = \ddot{y} = \mathcal{L}_f^2(h)$$

$\vdots$

$$\xi_{r_1-2} = \dot{\xi}_{r_1-1} = y^{[r_1-1]} = \mathcal{L}_f^{r_1-1}(h)$$

$$\xi_{r_1-1} = \dot{\xi}_{r_1} = \mathcal{L}_f^{r_1}(h) + \mathcal{L}_g(\mathcal{L}_f^{r_1-1}(h))u(t)$$

Let  $v = y^{[r]}$ , then

$$u = \frac{v - \mathcal{L}_f^{r_1}(h)}{\mathcal{L}_g(\mathcal{L}_f^{r_1-1}(h))} = \frac{v - \alpha_1(\mathbf{x})}{\beta_1(\mathbf{x})} \quad \forall \mathbf{x} \notin \mathcal{S}_1 \quad (28)$$

where,  $\mathcal{S}_1$  is the singularity space for  $r_1$ , defined as:

$$\mathcal{S}_1 = \left\{ \mathbf{x} \in \mathcal{R}^n \mid \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h(\mathbf{x}))) = 0 \right\}$$

When  $\mathbf{x} = \mathbf{x}_s \in \mathcal{S}_1$ , then  $\mathbf{x}_s$  is a singularity and the relative degree is not well defined and exact linearization will fail.

However, differentiating the output one more step yields:

$$\begin{aligned} \nu y^{[r_1+1]} &= \frac{d}{dt} \left[ \mathcal{L}_{\mathbf{f}}^{r_1}(h(\mathbf{x})) + \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h(\mathbf{x})))u(t) \right] \\ &= \mathcal{L}_{\mathbf{f}}^{r_1+1}(h(\mathbf{x})) \\ &+ \left[ \mathcal{L}_g^2(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h(\mathbf{x}))) \right] u^2 \\ &+ \left[ \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1}(h(\mathbf{x}))) + \mathcal{L}_{\mathbf{f}}(\mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h(\mathbf{x})))) \right] u \\ &+ \left[ \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h(\mathbf{x}))) \right] \dot{u} \end{aligned} \quad (29)$$

Where  $\mathbf{x} \in \mathcal{S}_1$ , then  $\mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h(\mathbf{x}))) = 0$ , Eq. (29) becomes

$$\begin{aligned} y^{[r_1+1]} &= \mathcal{L}_{\mathbf{f}}^{r_1+1}(h) + \left[ \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1}(h)) + \mathcal{L}_{\mathbf{f}}(\mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h))) \right] u \\ &+ \left[ \mathcal{L}_g^2(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h)) \right] u^2 \end{aligned}$$

and the system loses relative order. Let  $y^{[r_1+1]} = w$ , we have:

$$\begin{aligned} \left[ \mathcal{L}_g^2(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h)) \right] u^2 + \left[ \mathcal{L}_{\mathbf{f}}(\mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h))) + \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1}(h)) \right] u \\ + \left[ \mathcal{L}_{\mathbf{f}}^{r_1+1}(h) - w \right] = 0 \end{aligned} \quad (30)$$

which is quadratic in  $u$ . Generally this equation has two complex conjugate roots. However, if  $\mathcal{L}_g^2(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h)) = 0$ , then Eq. (30) becomes a linear equation in  $u$  with a real solution.

On the other hand if both  $\mathcal{L}_g^2(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h)) = 0$  and

$$\mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1}(h)) + \mathcal{L}_{\mathbf{f}}(\mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_1-1}(h))) = 0 \quad \forall \mathbf{x} \in \mathcal{Z}_{r_1} \quad (31)$$

then, Eq. (29) needs to be differentiated further until the next (secondary) relative order  $r_2$ , where

$$\begin{aligned} \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^k(h)) &= 0 \quad r_1 < k < r_2 - 2 \\ \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_2-1}(h)) &\neq 0 \quad \forall \mathbf{x} \notin \mathcal{S}_2 \subset \mathcal{R}^n \end{aligned}$$

where  $\mathcal{S}_2$  is the singularity space for  $r_2$ , defined as:

$$\mathcal{S}_2 = \left\{ \mathbf{x} \in \mathcal{S}_1 \cup \mathcal{Z}_1 \mid \mathcal{L}_g(\mathcal{L}_{\mathbf{f}}^{r_2-1}(h)) = 0 \right\}$$

This means that when  $\mathbf{x} \in \mathcal{S}_1$ , and  $r_2 < n$  then, through  $u$  we can control the  $r_2^{\text{th}}$  derivative of the  $y$ .

A similar procedure as for  $r_1$  can be used to design the FBL or SMC. A switching controller can be designed that uses an  $r^{\text{th}}$  controller (28) when the system is away from singularity  $\mathcal{S}_1$ , and switches to two  $(r_2)^{\text{th}}$  controller when  $\mathbf{x} \in \mathcal{S}_1 \cup \mathcal{Z}_1$ . On the other hand, if  $\mathbf{x} \in \mathcal{S}_1 \cup \mathcal{S}_2$ , the procedure could be used recursively  $k$ -times until  $r_k = n$  after which the system becomes uncontrollable.

## 8. CONCLUSIONS

This paper presented an approach to feedback linearization(FBL) and sliding mode control(SMC) of a class of non-regular systems. As in SMC and FBL, the output is differentiated  $r_1$  times till the control variable  $u$  appears. The coefficient that multiplies  $u$  vanishes in a certain subspace  $\mathcal{S}_1(\mathbf{x})$ . In this region, traditional FBL, and SMC techniques fail. Our claim is that, by further differentiating the output, a nonlinear (polynomial) differential equation of  $u$  is obtained. At the singularity region,  $\mathcal{S}_1$ , the coefficient of the  $\dot{u}$  term disappears and the differential equation degenerates to a quadratic equation. If the quadratic equation has only real roots, the system has a well defined relative degree at the singularity equal to  $r_1 + 1$ .

Switching controllers can be designed to switch from an  $r_1^{\text{th}}$  order controller to two  $(r_1 + 1)^{\text{th}}$  order controllers when the system is in the neighborhood of the singularity, or even to a  $r_2^{\text{th}}$  order controller in the singularity space.

Further research will focus on the condition under which the quadratic equation will have only real solutions and the relationship between the type of the roots and the stabilizability of the system. An adaptive switching condition is also under study.

## REFERENCES

- C. Barbu, R. Sepulchre, W. Lin, and P.V. Kotovic. Global asymptotic stabilization of the ball-and-beam system. In *Proceedings of the 35th IEEE Conference on Decision and Control*, pages 2351–2355, Dec 1997.
- B. Fernández-Rodríguez. *Sliding Model Control of Multi-Input Multi-Output Systems*. PhD dissertation, Massachusetts Institute of Technology, 1998.
- J. Hauser, S. Sastry, and P. Kokotović. Nonlinear control via approximate input-output linearization: the ball and beam example. *IEEE Transactions on Automatic Control*, 37(3):392–398, March 1992.
- M.C. Lai, C.C. Chien, C.Y. Cheng, Z. Xu, , and Y. Zhang. Nonlinear tracking control via approximate backstepping. In *American Control Conference*, volume 2, pages 1339–1343, Jun 1994.
- D.J. Leith and W.E. Leithead. Input-output linearisation of nonlinear systems with ill-defined relative degree: the ball and beam revisited. In *American Control Conference*, volume 4, pages 2811–2816, 2001.
- M.A. Marra, B.E. Boling, and B.L. Walcott. Genetic control of a ball-beam system. In *Proceedings of the 1996 IEEE International Conference on Control Applications*, pages 608–613, Sep 1996.
- Using Simulink® 6*. The MathWorks, Inc., Natick MA, 2004.
- C. Tomlin and S. Sastry. Switching through singularities. In *Proceedings of the 36th IEEE Conference on Decision and Control*, volume 1, pages 1–6, 1997.
- G. Yi, D.J. Hill, , and Z.-P. Jiang. Global nonlinear control of the ball and beam system. In *Proceedings of the 35th IEEE Conference on Decision and Control*, volume 3, pages 2818–2823, Dec 1996.