Introductory Talk on DDE's

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Abstract

This paper is based on the notes I used to give a talk on the theory of Delay Differential Equations (DDE) to Prakash Panangaden and Hans Vangheluwe for the School of Computer Science of McGill University on May 15, 2001. I filled in the details I had left out during the talk, and I have added several subjects (for completeness) as section 2.5, section 4 and appendix A; section 5 is quite more general than in my talk and section 7 has been completely changed because I acquired new information during this summer.

Contents

1 Heuristic Definition and Examples

1.1 Definition

We'll say that a ***Delay Differential Equation*** (DDE) is a differential equation in which the arguments of the unknown function can be *delayed* with respect to the one of the *highest order derivative* of this function (which is *not* delayed).

A format of DDE general enough for the 3 first sections is the following:

$$\vec{y}\,'(t) = \vec{f}\Big(t, \vec{y}\big(t - \tau_1(t)\big), \ldots, \vec{y}\big(t - \tau_m(t)\big)\Big) \tag{1.1}$$

where the $\tau_j(t)$'s are positive functions of t (the *delays*), $\vec{y}(t) : [t_0 - r, b] \to \mathbb{R}^n$[1] and $\vec{f} : [t_0 - r, b] \times \mathbb{R}^{n \times m} \to \mathbb{R}^n$ are vector-valued functions.

1.2 Examples

You may find applications of DDE's in:

- mixing liquids (it takes a certain time before the liquids are mixed)
- logistic model; e.g. $N'(t) = k[1 - \frac{N(t-r)}{P}]N(t)$ where the delay r takes in consideration the time before obtaining the effect of shortage of food, competition, etc.; N is the number of beings, P is the population constant and k is some proportionality constant
- 2-body force interaction (electromagnetic theory, for example, because light propagates at finite speed c)
- **control systems** (main field where DDE's occur) as in servomechanisms, neurological control models, etc.

2 Some Observations

To get a feel of how different DDE's are from ODE's (even if an ODE is a special case of a DDE), here are some observations for a linear homogeneous scalar DDE with one constant delay (which is quite less exotic than what a DDE could be!):

$$\frac{dy(t)}{dt} = A(t)y(t) + B(t)y(t - r) \tag{2.1}$$

where $A(t)$ and $B(t)$ are scalar real functions.

[1] The meaning of r will be explained in subsection 2.2.

2.1 Behaviour

Even very simple DDE's can have a quite different behaviour than ODE. For example, by considering $y'(t) = -y(t - \pi/2)$, we see that a first order scalar linear homogenous DDE with real coefficients can have oscillating solutions (try $cos(t)$ or $sin(t)$ for example), whereas the equivalent (first order scalar linear homogeneous) ODE with real coefficients *never* has oscillating solutions...

2.2 Uniqueness Condition

If a solution to equation (2.1) exists for $t \in]-\infty, t_0]$ or $[t_0, \infty[$, it will be in general *non-unique* with $y(t_0) = y_0$ specified. Even if you specify the derivatives of *all* orders of $y(t)$ at one point in time, the solution won't necessarily be unique!

For example, we consider equation (2.1) with the following condition:

$$y(t_0) = y_{00}, \qquad y'(t_0) = y_{01}, \qquad \ldots \qquad y^{(n)}(t_0) = y_{0n}, \qquad \ldots \tag{2.2}$$

We take $t_0 = 0$ for simplicity and consider the following function on $[-r, 0]$:

$$\theta(t) = \begin{cases} 0 & \text{for} \quad t = -r \\ e^{-t^2} e^{-(t+r)^{-2}} & \text{for} \quad -r < t < 0 \\ 0 & \text{for} \quad t = 0 \end{cases} \tag{2.3}$$

This is a C^∞ function with all derivatives equal to zero at $t = 0$ (check this!). If we assume that $A(t)$ and $B(t)$ are continuous, we can extend $\theta(t)$ on $[0, \infty[$ to a function which satisfies equation (2.1) by using the method of steps (see section 3). Call this function $x(t)$. Now, if there exists a function $y(t)$ which is a solution to equation (2.1) with condition (2.2), you can check that $z(t) = y(t) + cx(t)$ is also a solution to this problem, for any c! If a solution exists to (2.1) and (2.2), an infinite number of solutions exists!

As we'll see in section 5, to have a unique solution to a DDE like (1.1) one needs not an initial condition, as for ODE's, but an *initial function* on $[t_0 - r, t_0]$, where r is a bound big enough so that $t_0 - r \leq t - \tau_j(t)$ for all j's and for $t \in [t_0, b]$. This means that instead of working on a finite dimensional vector space (as in linear ODE's, where the general solution to a n$^{\text{th}}$ order scalar ODE can be expressed as a linear combination of n linearly independent solutions), we work here with an uncountably infinite dimensional vector space to characterize fully the general solution to a DDE! This makes it very hard to characterize it analytically...

2.3 Causality

Time now has a special direction (forward direction): no *backward* solution to (2.1) exists in general, even if only $x(t_0)$ is specified; and if by chance a backward solution to (2.1) exists, it won't be unique in general even if an initial function is specified on $[t_0 - r, t_0]$. This means that the backward problem is quite ill-conditioned. This shows pretty well the causality that is intrinsic in

DDE, whereas ODE can be continued in the past or in the future without much difference. Moreover, we consider in general the derivative in a DDE to represent a right-hand derivative instead of a total derivative, to take into account this special direction of time (and this also permits us to work with a discontinuous DE or initial function).

2.4 MacLaurin approximation

Solutions to (2.1) may behave quite differently than the solutions of the approximated problem of replacing (2.1) by an "approximated" ODE using a MacLaurin series in powers of r, that is, using:

$$y(t-r) = y(t) - ry'(t) + \frac{1}{2}r^2y''(t) - \cdots + \frac{(-1)^m}{m!}r^my^{(m)}(t) + \cdots$$

and truncating the series after a few terms, considering r to be small...

For example, one can prove that all solutions of

$$y'(t) = -2y(t) + y(t-r)$$

are bounded as $t \to \infty$. However, one can prove that the ODE you obtain by truncating the MacLaurin series after the third term:

$$y'(t) = -2y(t) + [y(t) - ry'(t) + \frac{1}{2}r^2y''(t)]$$

has exponentially increasing solutions, $ce^{\lambda t}$ with $\lambda > 0$ regardless of how small the value of $r > 0$.

2.5 Smoothness

The smoothness relationship between the DDE and the solutions is different than for an ODE. By implicitly differentiating both sides of the ODE:

$$y'(t) = f\big(t, y(t)\big)$$

we obtain that if f is C^∞, then the solution $y(t)$ is also C^∞. On the other hand, even if $\vec{f}$, $\tau_j(t)$'s in (1.1) and its initial function are C^∞ (considering right-hand derivatives), the solution is not even necessarily C^1! This is because the initial function doesn't satisfy the DDE in general, which causes a discontinuity in the modelling of the phenomenon and gives rise to non-smooth solutions. More details about this phenomenon can be found in [7] and [5].

3 Method of Steps

This is the first analytical method that is always presented to solve a DDE. Considering equation (1.1), it can be used whenever

$$0 < s \leq \inf_{t\in[t_0,b]} \tau_j(t)$$

for all j 's except maybe one which needs then to be always equal to 0 (the ODE component). To get a clearer idea, consider the method on the following simpler DDE:

$$y'(t) = f\Big(t, y(t), y\big(t - \tau(t)\big)\Big) \tag{3.1}$$

with $0 < s \leq \tau(t)$ and a continuous initial function $\theta(t)$ specified on $[t_0 - r, t]$, where r is big enough so that $t_0 - r \leq t - \tau(t)$ for $t \in [t_0, b]$. Then because you know the initial function, you know the values of $y(t - \tau(t))$ for $t \in [t_0, t_0 + s]$ since $s \leq \tau(t)$. This means that you can substitute this function in f to obtain a simple ODE to solve on $[t_0, t_0 + s]$. To guarantee a solution, you would need some conditions on f (for example, f continuous and Lipschitzian in its second variable would be enough). Then, you can use this new solution to find out the values of $y(t - \tau(t))$ for $t \in [t_0 + s, t_0 + 2s]$ and thus solve the resulting ODE on $[t_0 + s, t_0 + 2s]$. You may continue the solution of the DDE similarly with steps of time of length s as far as you want, as long as $s \leq \tau(t)$ and that a solution to the resulting ODE's exists.

3.1 Key Fact

What is very interesting is that if you don't have any ODE component, i. e. all the delays are always greater than 0, then you need *very weak* conditions on f to guarantee the existence of a unique solution to the DDE; conditions a lot weaker than you would need for an ODE! To see that, integrate both sides of equation (3.1) and use the fundamental theorem of calculus to obtain the following:

$$y(t) = y(t_0) + \int_{t_0}^{t} f\Big(\xi, y\big(\xi - \tau(\xi)\big)\Big) d\xi$$

Since $y(t - \tau(t)) = \phi(t - \tau(t))$ (the initial function; see theorem 1) for $t \in [t_0, t_0 + s]$, this expression gives the unique solution to the DDE on $[t_0, t_0 + s]$ if we can use the fundamental theorem of calculus there. The continuity of f, ϕ and τ are enough to guarantee the continuity of the integrand, and hence the validity of the fundamental theorem of calculus there. This means that *mere continuity conditions guarantee the existence of a unique solution to the DDE when all the delays are greater than 0*! This is a far weaker condition than the Lipschitz condition on f that you normally ask to guarantee existence of a solution for an ODE. This also means that for any continuous ODE, the fact of replacing $y(t)$ by $y(t - r)$ guarantees the existence of a global solution to the corresponding DDE, and with as small r as you wish! Try this on the ODE: $y'(t) = y^2(t)$ for example and see how nicer is the solution to the corresponding DDE: $y'(t) = y^2(t - r)$ (see also section 5.7)! By the way, this result is what pushed Prakash to make me study DDE's this summer...

4 Notation for RFDE

One can generalize theorems from ODE's to DDE's quite easily with proper notation. Indeed, apart the key fact mentioned in section 3.1, the uniqueness and existence theorems which will be presented in section 5 are almost exactly the same as the ones for ODE's. But this is quite more transparent if we consider more general DDE's than equation (1.1), that is, if we look at *Retarded Functional Differential Equations* (RFDE) instead (see appendix A which explains my terminology).

Hence, we present here a conventional notation from Shimanov.

Given a function $\vec{x}$ defined at least on $[t-r, t] \to \mathbb{R}^n$, we define a new function $\vec{x}_t : [-r, 0] \to \mathbb{R}^n$ by

$$\vec{x}_t(\sigma) = \vec{x}(t+\sigma) \quad \text{for} \quad -r \leq \sigma \leq 0.$$

Note that $\vec{x}_t$ is obtained by considering only $\vec{x}(s)$ for $t - r \leq s \leq t$ and then translating this segment of $\vec{x}$ to the interval $[-r, 0]$ which permits us to work with a fixed interval for our functionals...

We then define $\mathcal{C}_D$ as $C([-r, 0], D)$, where r is some positive real number[2], D is some *open* subset of $\mathbb{R}^n$ and, as usual, $C([-r, 0], D)$ is the set of continuous function from $[-r, 0]$ to D with the Euclidean topology. Finally, we make $\mathcal{C}_D$ as a Banach space by considering the sup norm $||\cdot||_r$ on it.

We're now ready to introduce the RFDE. Let J denote the interval $[t_0, b]$. Then we consider the *functional*[3]

$$\vec{F} : J \times \mathcal{C}_D \to \mathbb{R}^n \tag{4.1}$$

which maps a point of time and a continuous function on $[-r, 0]$ to a point in $\mathbb{R}^n$ (it's a function of functions!). The differential equation:

$$\vec{y}\,'(t) = F(t, \vec{y}_t) \tag{4.2}$$

represents truly a RFDE. And the initial function condition is prescribed by:

$$\vec{y}_{t_0} = \vec{\phi} \tag{4.3}$$

For example, by defining $F(t, \psi) \equiv \int_{-r}^{0} \psi(s)ds$, equation (4.2) becomes exactly equivalent to equation (A.2) (see appendix A).

Let's see now how to define the functional $\vec{F}$ to obtain exactly our original DDE (1.1). So given (1.1) with bounded delays by r (i. e. $0 \leq \tau_j(t) \leq r$), we define a new functional

$$\vec{F}(t, \vec{\psi}) \equiv \vec{f}\big(t, \vec{\psi}(-\tau_1(t)), \ldots, \vec{\psi}(-\tau_m(t))\big) \tag{4.4}$$

By substituting the functional (4.4) in the RFDE (4.2) and interpreting properly the notation $\vec{y}_t$, we obtain exactly the DDE (1.1).

[2] In a completely general treatment, r could be ∞, but we consider in this paper only bounded delays, i. e. $r < \infty$.

[3] The functional defined by (4.1) is sometimes called a ***Volterra functional*** because it uses only *past* values of the function (see [3]).

5 Existence and Uniqueness Theorems

For all this section, we consider $\vec{F}$ to be a functional $\vec{F} : J \times \mathcal{C}_D \to \mathbb{R}^n$ where $J \equiv [t_0, \beta[$ and D is some open subset of $\mathbb{R}^n$. All the proofs of these theorems can be found in Driver [4].

Before stating the theorems, we need three definitions[4].

5.1 Some Definitions

Definition 1 (weak continuous) *The functional $\vec{F}$ is said to be* **weak continuous** *(or* **weak C***) iff $\vec{F}(t, \vec{x}_t)$ is continuous with respect to t in J for each given continuous function $\vec{x} : J \to D$.*

Definition 2 (locally Lipschitzian) *The functional $\vec{F}$ is said to be* **locally Lipschitzian** with respect to $\mathcal{C}_D$ *iff for each $(t, \vec{\phi})$ in $J \times \mathcal{C}_D$, there exists a neighborhood $\mathcal{B}$ of $(t, \vec{\phi})$ in $J \times \mathcal{C}_D$ such that $\vec{F}$ is* Lipschitzian *on $\mathcal{B}$ with respect to $\mathcal{C}_D$ (i. e. there exists a positive constant K such that $||\vec{F}(t, \vec{\psi}) - \vec{F}(t, \vec{\phi})|| \leq K||\vec{\psi} - \vec{\phi}||_r$ whenever $(t, \vec{\psi})$ and $(t, \vec{\phi})$ are in $\mathcal{B}$.*

Definition 3 (quasi-bounded) *The functional $\vec{F}$ is said to be* **quasi-bounded** *iff $\vec{F}$ is bounded on every set of the form $[t_0, \beta_1] \times \mathcal{C}_A$ where $t_0 < \beta_1 < \beta$ and A is a compact[5] subset of D.*

Let's see the relationship between those definitions and our usual DDE (1.1). For definition 2, note that if $\vec{f}$ in (1.1) is locally Lipschitzian[6] with respect to all but its first argument, then $\vec{F}$ defined by (4.4) is locally Lipschitzian with respect to its second argument. For definition 1, $\vec{f}$ and $\tau_j(t)$'s being continuous is sufficient to guarantee the continuity of $\vec{F}$ and hence, its weak continuity since one can easily check that continuity implies weak continuity in our definition.

We're now ready to state the main uniqueness and existence theorems for RFDE's. If you're well acquainted with uniqueness and existence theorems for ODE's, you will recognize that the ones for RFDE's are pretty much the same...

5.2 Uniqueness

Theorem 1 (uniqueness) *If $\vec{F}$ is* weak C *and* locally Lipschitzian *with respect to $\mathcal{C}_D$, then given any $\vec{\phi} \in \mathcal{C}_D$, the RFDE (4.2) with initial function $\vec{y}_{t_0} = \vec{\phi}$ has* at most *one solution on $[t_0 - r, \beta_1[$ for any $\beta_1 \in]t_0, \beta]$.*

This theorem looks almost exactly the same for ODE's: the only difference is that the concept of 'weak continuity' doesn't make sense for ODE's... Anyhow,

[4]The name 'weak continuous' is from me. I saw this concept introduced only in Driver [4] up to now, so I took the liberty to give it a name...

[5]In this case, it means A is closed and bounded since D is a subset of $\mathbb{R}^n$

[6]$\vec{f}$ being C^1 is a sufficient (but not necessary) condition to be locally Lipschitzian.

since continuity implies weak C, you may replace the weak C condition by a continuity condition without changing the result (except that you obtain a weaker (less general) theorem). The other advantage of weak C over normal continuity is that it is normally easier to check.

5.3 Continuous Dependance on Initial Function

Theorem 2 (continuous dependance) *Assume $\vec{F}$ is* weak C *and (globally)* Lipschitzian *with respect to $\mathcal{C}_D$ with Lipschitz constant K. Let ϕ & $\tilde{\phi} \in \mathcal{C}_D$ be given. Suppose y & $\tilde{y}$ are respectively the unique solution to the RFDE (4.2) with initial function $y_{t_0} = \phi$ & $\tilde{y}_{t_0} = \tilde{\phi}$ respectively. Then if y & $\tilde{y}$ are both valid on $[t_0 - r, \beta_1[$ for some $\beta_1 \in]t_0, \beta]$, we have that*

$$||y(t) - \tilde{y}(t)|| \leq ||\phi - \tilde{\phi}||_r e^{K(t-t_0)} \quad \textit{for } t_0 \leq t < \beta_1.$$

Again, you have exactly the same theorem for ODE's, with the sup norm of the difference of the initial functions replaced by the norm of the difference of the initial conditions.

5.4 Local Existence

Theorem 3 (local existence) *If $\vec{F}$ is* weak C *and* locally Lipschitzian *with respect to $\mathcal{C}_D$, then given any $\vec{\phi} \in \mathcal{C}_D$, there exists a $\Delta > 0$ such that a solution exists on $[t_0 - r, t_0 + \Delta[$ to the RFDE (4.2) with initial function $\vec{y}_{t_0} = \vec{\phi}$ (and this solution is unique by theorem 1).*

Here, there is a subtle but important difference with ODE's: theorem 3 states *forward* local existence whereas the equivalent theorem for ODE's (with the function continuous and locally Lipschitzian) states local existence on *both sides* (on $]t_0 - \Delta, t_0 + \Delta[$). As mentioned in section 2.3, this is because of the concept of causality embedded in DDE's.

5.5 Extended Existence

Theorem 4 (extended existence) *If $\vec{F}$ satisfies conditions of theorem 3 and in addition is* quasi-bounded*, then there exists a unique* noncontinuable[7] *solution $\vec{y}$ on $[t_0 - r, \beta_1[$ to the RFDE of theorem 3; and if $\beta_1 < \beta$, then for any closed bounded set $A \subseteq D$, $\vec{y}(t) \notin A$ for some $t \in]t_0, \beta_1[$.*

The corresponding theorem for ODE's is the same except that it doesn't need the condition of quasi-boundedness. This comes from the fact that a continuous function is necessarily bounded on a closed bounded interval, whereas a continuous functional is *not* necessarily bounded on a closed bounded subset $\mathcal{C}_A \subseteq \mathcal{C}_D$, since $\mathcal{C}_D$ is infinite dimensional, hence its *closed bounded* subsets are not necessarily *compact*...

[7] This means that there is no solution to the DE which is defined on a bigger interval.

The last statement in theorem 4 means that if you can prove by whatever mean that the solution of the RFDE problem stays in some bounded subset of the domain of the functional $\vec{F}$, then you have *global existence* since it implies that $\beta_1 = \beta$. In physics, one can often use energy methods to check this condition.

5.6 Global Existence

Using theorem 4, we can easily deduce the following global existence theorem.

Theorem 5 (global existence) *Let $D = \mathbb{R}^n$. Assume that $\vec{F}$ satisfies conditions of theorem 3 (local existence). Assume further that*

$$||\vec{F}(t,\vec{\phi})|| \leq M(t) + N(t)||\vec{\phi}||_r \tag{5.1}$$

on $[t_0, \beta[\times \mathcal{C}_{\mathbb{R}^n}$, where M and N are continuous positive functions on $[t_0, \beta[$. Then the unique non-continuable solution to the RFDE problem of theorem 3 exists on the entire *interval $[t_0 - r, \beta[$.*

Apart the subtle difference with direction of existence, the global existence theorem for ODE's is exactly the same (with some Euclidean norm replacing the sup norm).

We can now use theorem 5 to derive the two following corollaries which are, again, very similar to the ones for ODE's.

Corollary 6 (linear DDE) *For the linear DDE*

$$\vec{y}\,'(t) = \sum_{j=1}^{m} A_j(t)\vec{y}(t - \tau_j(t)) + \vec{h}(t) \qquad t \in [t_0, \beta[$$

where A_j's are continuous $n \times n$ matrix valued functions, h is a continuous n-vector valued function and g_j's are continuous real valued functions with $0 \leq \tau_j(t) \leq r$, theorem 5 applies (with $D = \mathbb{R}^n$) hence it has a global unique solution.

Corollary 7 (Lipschitz) *If $\vec{F}$* defined for *$D = \mathbb{R}^n$ is* weak C *and* (globally) Lipschitzian *with respect to $\mathcal{C}_{\mathbb{R}^n}$, then theorem 5 applies and there exists a global unique solution to the RFDE problem in theorem 3.*

5.7 Remarks

To summarize this section, we can conclude that with proper notation, the uniqueness and existence theorems for RFDE's are almost the same as for ODE's. There is one very important exception, though, which has been presented in section 3.1: when the delays are all greater than zero, mere continuity conditions guarantee the existence of a *global* unique solution! There is nothing similar to that for ODE's... Also, this theorem can provide us with some examples to show that the conditions stated in the above theorems were sufficient, but *not necessary*. For example, the very simple DDE

$$y'(t) = y^2(t-h)$$

doesn't satisfy condition (5.1) in theorem 5 (global existence), but the method of steps guarantee the existence of a global unique solution anyway!

6 Some Comments about Linear DDE's

I present here some general comments about linear DDE's. We consider the linear DDE presented in corollary 6:

$$\vec{y}\,'(t) = \sum_{j=1}^{m} A_j(t)\vec{y}(t-\tau_j(t)) + \vec{h}(t) \qquad t \in [t_0, \beta[\tag{6.1}$$

with proper continuity conditions which guarantee the existence of a unique solution on the whole interval $[t_0, \beta[$.

The fundamental principle of *superposition* obviously carries over to linear DDE's, as well as the tools of identifying a *particular solution* and a *homogeneous solution*.

Often, one can show that the homogenous solution to some linear DDE goes to 0 as t goes to infinity (considering $\beta = \infty$). Then the homogeneous solution is called the *transient solution* and the particular solution is the *steady-state solution*, which never depends on the initial conditions (as for ODE's). This is particularly useful since finding the general solution of a linear DDE is extremely difficult, as already mentioned in section 2.2. Even in the constant coefficients case, equation (6.1) has in general an *infinite* number of linearly independent solutions (mainly because $\mathcal{C}_D$ is infinite dimensional).

The best we can do in general is to study the long term behaviour of the solution (hopefully, this is what control engineers want to know most of the time; they want to know if the solutions are bounded, stable, etc.). For the constant coefficients case, one tries $\vec{x}(t) = e^{\lambda t}\vec{a}$ (where $\vec{a}$ is a constant vector) in equation (6.1) and obtain a transcendental equation in λ as a condition to obtain a non-trivial solution. This equation is called the ***characteristic equation*** for the DDE, a similar concept as in ODE's except it's not a polynomial in λ but a *quasi polynomial* (or *exponential polynomial*) which has in general an infinite number of roots. The position of those roots in the complex plane can inform us about the stability and boundedness of the solution. You can find in [2] a thorough introduction to this concept for DDE's.

Other concepts introduced in [2] to solve DDE's are a kind of Fourier expansions, and in depth use of Laplace transforms.

7 Numerical Solutions

Can we generalize to DDE's the numerical methods already used to solve ODE's? First of all, Euler's method with simple linear interpolation can solve any well-behaved RFDE (see [3]). To solve a DDE in general, interpolation is needed since sometimes points of the solution which weren't computed are needed to evaluate $\vec{F}$ at later points. This makes the error analysis a bit different than for ODE's. But still, the numerical methods for ODE's can sometimes apply to DDE's by simply adding an interpolation scheme (see [1]).

In [7], I study the possibility of using the method of steps numerically to solve DDE's. One can transform the DDE in a ODE at each step, and thus can use normal numerical methods for ODE's to solve the equation at each step (but still need to use interpolation for the transformation to be done). I prove in this paper that, with some smoothness assumptions on the equation, the global error of this method for the DDE is of the same order than the global error made at each step with ODE methods. Finer error analysis is still needed to be done to be able to use adaptive step size algorithms with this method.

Finally, I can say that a lot has been said in the literature about numerical solutions of DDE's, but I have not seen *yet* a systematic approach as there is for ODE's (like the general class of one-step methods with the concept of local truncation error). But still, there seems to exist algorithms which are said to solve very general class of problems (but I don't know yet how they really work) as in the appendix of [6] where an algorithm in Fortran is given and said to solve SNDDE (System of Neutral Delay Differential Equations[8]) with *state-dependent delays*[9]! It is said to be "a variable-step and variable-order code (...) based on Adams methods implemented in PEICEI (Predict-Evaluate-Interpolate-Correct-Evaluate-Interpolate) mode with order p ranging from 1 to 12"[10]. Wow!

A Notes on Terminology

The terminology for Delay Differential Equations (DDE) and other kinds of Functional Differential Equations (FDE) is not fixed (from what I saw in literature). I will give some heuristic definitions to get an idea of the subject.

Let's say that a ***Functional Differential Equation*** is an equation where derivatives of a functions are related through *functionals* (functions of func-

[8]A Neutral Delay Differential Equation (NDDE) is a bit more general than DDE's: the highest order derivative in the equation can also have delays. This innocent looking difference makes the existence and uniqueness theory of NDDE's a lot more complicated than DDE's...

[9]This means that you can have equations like:

$$y'(t) = f\Big(t, y(t), y\big(t - \tau(t, y(t))\big)\Big)$$

[10][6], p. 377.

tions). For example:

$$y'(t) = \int_{t-1}^{t+1} y(s)ds \tag{A.1}$$

where the integral here is the functional (it is a function $F : \mathbb{R} \times C(\mathbb{R}) \to \mathbb{R}$, $F(t, y) = \int_{t-1}^{t+1} y(s)ds$, where $C(\mathbb{R})$ is the Banach space of continuous real functions).

DDE is a special case of FDE. We'll say that a ***Delay Differential Equation*** is a FDE where the arguments of the unknown function and its derivatives can be *delayed* with respect to the one of the *highest order derivative* of this function (which is *not* delayed). DDE is sometimes called a ***Retarded Functional Differential Equation*** (RFDE) or ***Differential-Difference Equation*** (also DDE; see [2]). Personnally, I would use RFDE to put emphasis on the functional aspect of the differential equation and use the name DDE only for differential equations with a finite number of delays. Hence RFDE would be more general than DDE. For example, DDE would mainly be of the form of equation (1.1) whereas RFDE could also include equations like

$$y'(t) = \int_{t-r}^{t} y(s)ds \tag{A.2}$$

which has an uncountable number of delays (and thus wouldn't be a DDE in my sense). Also, equation (A.1) would be a FDE and not a RFDE because future evaluations of y are also used...

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