Summer Report about DDE's

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September 3, 2001

Abstract

This paper gives a summary of the results I obtained while working on DDE's with Prakash Panangaden and Hans Vangheluwe for the School of Computer Science of McGill University this summer. It uses also my other paper, *Introductory Talk on DDE's* (see [9]), as both an introduction to DDE's and as a report about the theory of DDE's.

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1 Introduction

The purpose of studying DDE's this summer was to try to understand their *intrinsic* stability properties. It seems that they can have very general existence theorems (more general than ODE's) as we can see in the *Key Fact* section of [9]. The unsaid aim of Prakash (until late during the summer...) was to make a parallel with some very general fixed point existence theorems in dataflow computation. It could be the case that the method of step could be generalized to dataflow problems, or that there would exist a canonical general theory which could include those global existence theorems for DDE's and for dataflow fixed point problems as special cases... Maybe!

In between the study of existence and stability of DDE's, I worked a bit with Hans about the theory of numerical solutions to DDE's. Error bounds and convergence were studied. Unfortunately, I haven't finished the summer with some definite results about DDE's (as an efficient working algorithm for example, or a complete analysis of their convergence). The result consists mainly of a better understanding of DDE's plus a good experience of what fundamental research looks like...

Following is a summary of the results I obtained this summer.

2 Existence and Uniqueness theory

This has already been covered pretty well in [9].

3 Stability Theory (for Control Systems)

In control systems, we usually use the ***transfer function*** $P(s)$ to characterize the system. The transfer function of a system is the laplace transform of its *impulse response*, that is, the output (solution) to the system when all initial values are 0 and the forcing term (input) is the dirac-delta function. We then have the following general relationship in the s-domain:

$$\underbrace{Y(s)}_{\text{output}} = \underbrace{P(s)}_{\text{transfer}} \underbrace{X(s)}_{\text{input}}.$$

Also, the composition of components in the system is replaced by the multiplication of their respective transfer functions (this explains the usefulness of the transfer function). The inverse laplace transform of a rational function $P(s) = \left\{ \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k} \right\}$ has the following form (see [4]):

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k} \right\}(t) &= \mathcal{L}^{-1} \left\{ b_m + \sum_{k=1}^{r} \sum_{l=1}^{n_k} \frac{c_{kl}}{(s+p_k)^l} \right\}(t) \\ &= b_m \delta(t) + \sum_{k=1}^{r} \sum_{l=1}^{n_k} \frac{c_{kl}}{(l-1)!} t^{l-1} e^{-p_k t} \end{aligned}$$

where r is the number of different poles of $P(s)$, p_k's are the poles of $P(s)$, n_k is the multiplicity of the pole p_k and c_{kl}'s are the residues of $P(s)$. Normally, the degree m of the numerator is smaller than the degree n of the denominator, hence $b_m = 0$. By looking at this equation, we see that in order to obtain an impulse response $p(t)$ which goes to 0 as $t \to \infty$, we need *all the poles p_k to be in the left half complex plane* (i.e. they have negative real part so that the exponential terms decay to zero). We then say that the system is ***stable***, since the condition that the impulse response goes to zero as time approaches infinity is equivalent to the condition that every bounded input produces a bounded output for this system.

For a feedback delay system (with constant delay h for example), an exponential term e^{-hs} appears in its transfer function. This means that the characteristic equation of $P(s)$ is not a polynomial as usual, but a *quasi-polynomial* like $D(s) + e^{-hs}N(s)$ for example. But the stability of the system can still be related to the position of the roots of the characteristic equation in the complex plane. The **important conclusion** that we can draw here (for Prakash) is that the stability of the linear delay system with h very small *will be the same* as when $h = 0$ by a *continuity argument*, except if the system is marginally stable when $h = 0$ (which means that some roots are on the imaginary axis). This is because the position of the roots in the complex plane changes continuously as h changes, so that they will stay in the same region of stability if h is small enough, except if the roots were already at the limit of a region of stability (as the imaginary axis).

A few words about linear delay systems and ways to approximate them (using Padé approximations, for example) are given in [10]. A thorough presentation of how to locate the roots of the characteristic equation is given in [2].

4 Error Analysis and Numerical Solutions

4.1 Euler's Method

In [3], it is shown that Euler's method with linear interpolation converges for any RFDE which is *globally Lipschitzian* and *weak C*. It uses a mere generalization of Henrici's method (which proved convergence of Euler's method for ODE's in the first chapter of [7]) but with proper notation for Banach spaces. In this paper, it is also shown that if we assume that the solution to the RFDE is C^2 and that the error on the initial function is $O(h)$, where h is the step size of the method, then the global error of Euler's method is $O(h)$ (as for ODE's).

4.2 ODE-based Methods

In [8], I explain how we can use the method of steps to solve a DDE with positive delays, using numerical algorithms for ODE's successively with some interpolation scheme. I prove that, with non-trivial assumptions on the equation, the order of convergence of the global method is the same than the order of convergence of the ODE method at each step.

Hans' method to solve the DDE's was the following. Using the initial function, you can transform the DDE to an ODE on a small interval. Let's say now that we can use a normal numerical algorithms which solves ODE's to find an approximation to the solution to the DDE a step h further. With some interpolation, we can again use this approximated solution to transform the DDE to an ODE on another small interval. You repeat the same process, and this way you get an approximated solution to the DDE as far as you want. Theoretically, you could repeat the same arguments I used in [8] to prove that the order of convergence of this method (with some smoothness assumptions) is the same than the ODE algorithm you use. But what makes this far from being obvious is the following: the ODE we are trying to solve *changes with time*, because we are repeatedly feeding in the approximated solution in the functional to obtain a simple ODE.

For example, if we consider the simple DDE: $y'(t) = f(t, y(t), y(t-\tau))$, by using the approximated solution found at $t - \tau$, we can obtain a new simple ODE $y'(t) = g(t, y(t))$ on a small interval where $g(t, y(t)) = f(t, y(t), w(t-\tau))$ and w is the approximated solution for y. Because of this dynamic change in the ODE, we could ask ourselves if the order of convergence proofs of the ODE methods are still valid. For example, Taylor's expansion of $f(t, y(t))$ is used to prove the order of convergence of the Runge-Kutta method. But what would be a Taylor expansion of $f(t, y(t), y(t-\tau)$? Surprisingly, it is said in [1] that normal ODE methods can be used to solve DDE's with the *same order of convergence*, assuming you use some right interpolation scheme. Honestly, I have still doubts about this declaration (because of what I said above and because they don't give any clear example of how to implement their method). But unfortunately, I haven't had the time to study that in details and find either a counter-example or understand thoroughly the proof of it... So I can't say more about it for now!

References

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