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Mode Transition Behavior in Hybrid Dynamic Systems

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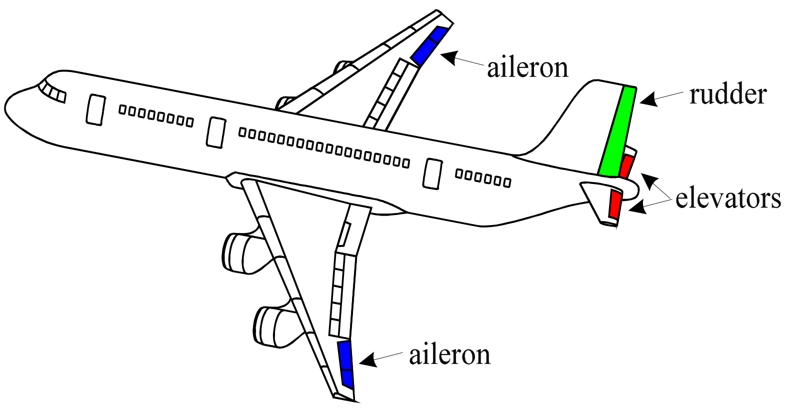
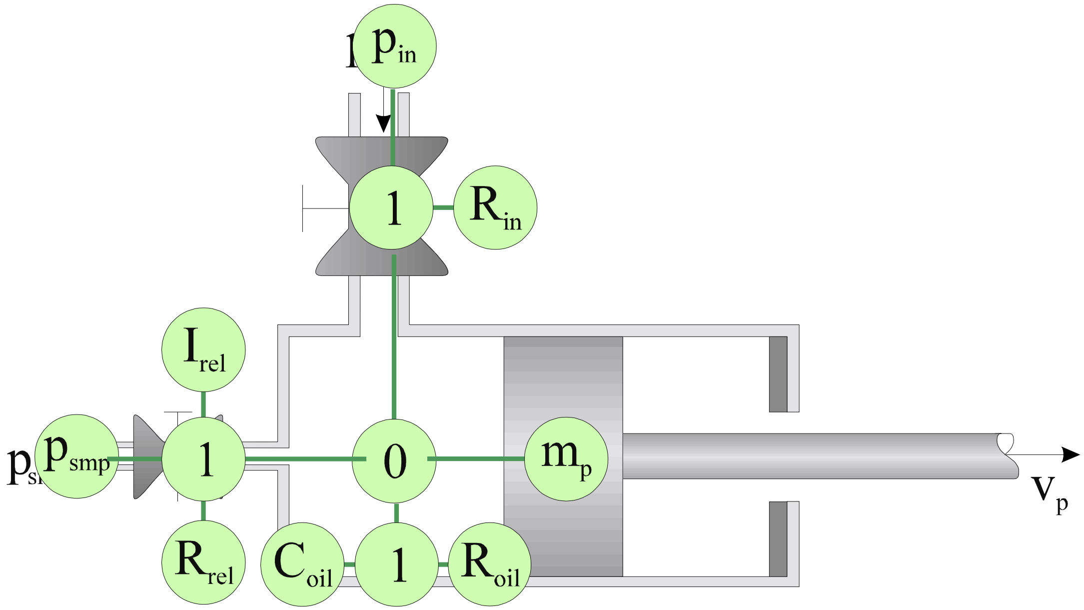
Introduction

- Mode Transitions in Hybrid Models of Physical Systems
  - hybrid because
    - continuous, differential equations
    - discrete, finite state machine
  - overview of phenomena involved
- Illustrated by Hydraulic Actuator Used for Aircraft Attitude Control Surfaces



Modeling of Physical Systems

- Ideal Picture Model (Schematic)
- Identify Behavioral Phenomena
- For Example, A Hydraulic Actuator



Equation Generation

- Compile Constituent Equations
  - R_{in} — $f_{in} R_{in} = p_{Rin}$
  - R_{oil} — $f_R R_{oil} = p_{Roil}$
  - C_{oil} — $C_{oil} \dot{p}_C = f_R$
  - m_p — $m_p \dot{v}_p = A_p p_{cyl}$
  - R_{rel} — $f_{rel} R_{rel} = p_{rel}$
  - I_{rel} — $I_{rel} \dot{f}_{rel} = p_{rel}$
  - *0*, cylinder chamber — $v_p = f_{in} - f_{rel}$
  - *1*, relief flow pipe — $p_{rel} = p_{smp} - f_{rel} R_{rel} + p_{cyl}$
  - *1*, intake pipe — $p_{Rin} = p_{in} - p_{cyl}$
  - *1*, oil compression — $p_{Roil} = p_{oil} - p_C$

Equation Processing

- Before Simulation
  - the number of equations is reduced
    - substitution/elimination
  - equations are sorted
    - each equation computes one variable
  - equations are solved
    - high index problems may require differentiation of certain equations

Hybrid Behavior

- Introduce Valves
  - make highly nonlinear behavior piecewise linear
    - intake valve

$$\text{if} \quad v_{in} \quad \text{then} \quad p_{Rin} = p_{in} - p_{cyl} \quad \text{else} \quad f_{in} = 0$$

    - relief valve

$$\text{if} \quad v_{rel} \quad \text{then} \quad p_{rel} = p_{smp} - f_{rel} R_{rel} + p_{cyl} \quad \text{else} \quad f_{rel} = 0$$

- Switching Between Modes of Continuous Behavior
  - intake valve, v_{in}, external switch (control law)
  - relief valve, v_{rel}, autonomous switch triggered by physical quantities

$$v_{rel} = p_{cyl} > p_{th}$$

  - different sets of equations

Computational Causality

- When Switching Equations
  - computational causality may change
    - re-ordering
    - re-solving
- Example
  - when the intake valve closes, equations change
    - From

$$p_{Rin} = p_{in} - p_{cyl}$$

    - To

$$f_{in} = 0$$

  - therefore, in this equation
    - p_{Rin} becomes unknown
    - f_{in} becomes known

Implicit Modeling

- Deal With Causal Changes Numerically
- Valve Behavior
  - residue on f_{in}

$$0 = if \quad v_{in} \quad then \quad -p_{Rin} + p_{in} - p_{cyl} \quad else \quad f_{in}$$

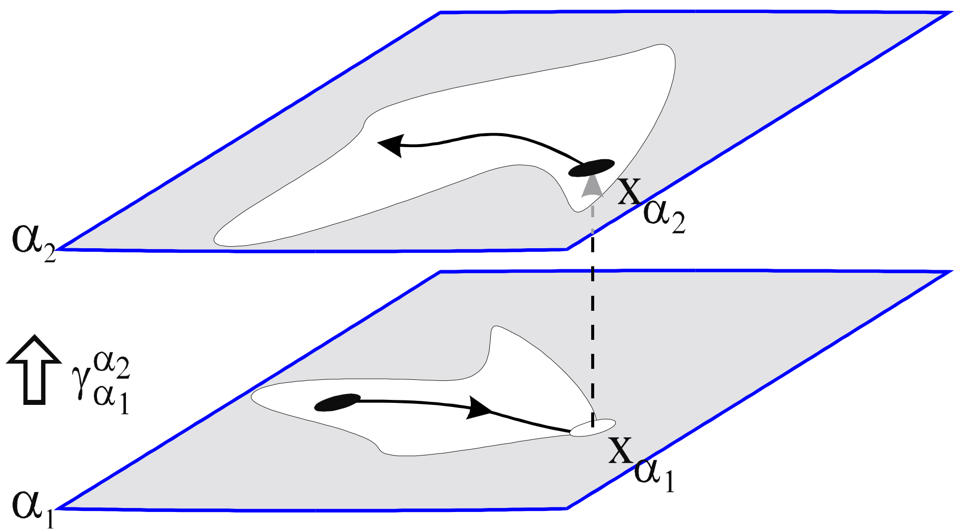
  - residue on f_{rel}

$$0 = if \quad v_{rel} \quad then \quad -p_{rel} + p_{smp} - f_{rel} R_{rel} + p_{cyl} \quad else \quad f_{rel}$$

- Implicit Numerical Solver (e.g., DASSL)
  - designed to handle this formulation

Hybrid Dynamic Behavior

- Geometric View
  - modes of continuous, smooth, behavior
  - patches of admissible state variable values



Specification Parts

- Hybrid Behavior Specification
  - a function, ***f***, that defines continuous, smooth, behavior for each mode

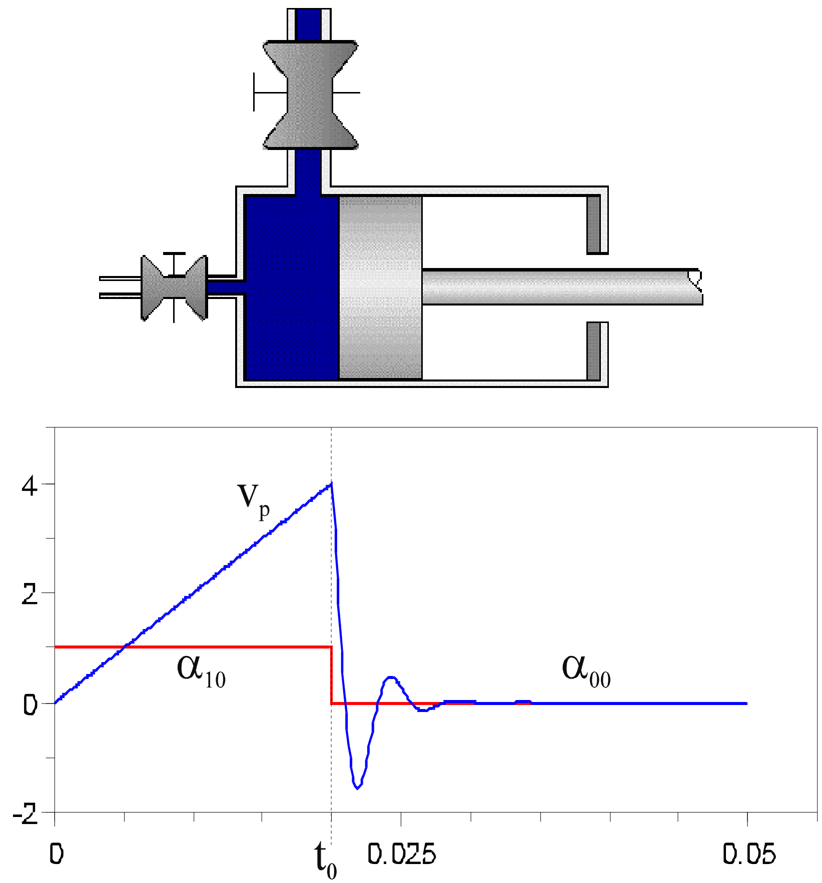
$$f_{\alpha_i}: E_{\alpha_i}\dot{x} + A_{\alpha_i}x + B_{\alpha_i}u = 0$$

  - an inequality, γ, that defines admissible state variable values

$$\gamma_{\alpha_i}^{\alpha_{i+1}}: C_{\alpha_i}x + D_{\alpha_i}u \geq 0$$

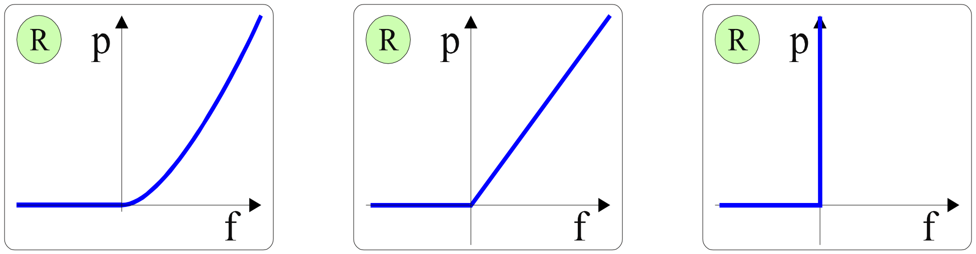
Dynamics

- Behavior Characteristics
  - C^0, i.e., no jumps in state variables
  - steep gradients
- Example
  - when the intake valve closes, piston velocity quickly reduces to 0



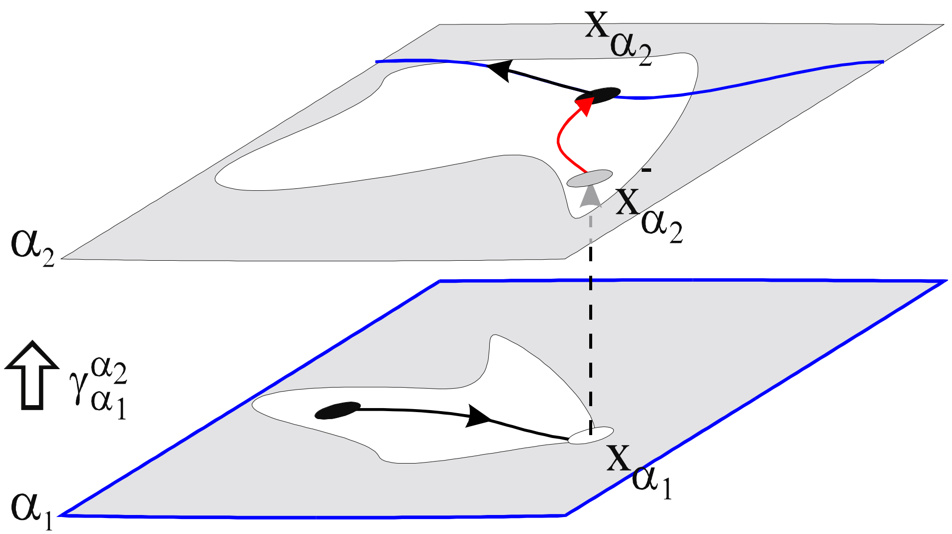
The Next Step

- Remove Steep Gradients
  - e.g., singular perturbation
- Algebraic Constraints Between State Variables
  - high index systems
  - subspace with admissible (continuous) dynamic behavior
  - discontinuities (jumps) in state behavior



Hybrid Dynamic Behavior - Refined

- Geometric View
  - modes of continuous, smooth, behavior
  - patches of admissible state variable values
  - manifold of dynamic behavior



Specification Parts

- Hybrid Behavior Specification
  - a function, f, that implicitly defines for each mode
    - continuous, smooth, behavior
    - state variable value jumps

$$f_{\alpha_i}: E_{\alpha_i}\dot{x} + A_{\alpha_i}x + B_{\alpha_i}u = 0$$

  - an inequality, γ, that defines admissible generalized state variable values

$$\gamma_{\alpha_i}^{\alpha_{i+1}}: C_{\alpha_i}x + D_{\alpha_i}u \geq 0$$

  - for explicit reinitialization (semantics of x^-)

$$f_{\alpha_i}: E_{\alpha_i}\dot{x} + A_{\alpha_i}x + B_{\alpha_i}^{u}u + B_{\alpha_i}^{x}x^- = 0$$

Handling of Systems With High Index

- DASSL Handles Index 2 Systems
  - implicit formulation for continuous behavior
- Requires Consistent Initial Conditions When Mode Changes Occur
  - compute from implicit formulation to make jump space (projection) explicit
  - for example, sequences of subspace iteration
    - space of dynamic behavior: $V^{n+1} = A^{-1} E V^n, V^0 = R^n$
    - jump space: $T^{n+1} = E^{-1} A T^n, T^0 = \{0\}$
  - or, decomposition in (pseudo) Kronecker Normal Form

Projections

- Linear Time Invariant Index 2 System
  - derive pseudo Kronecker Normal Form (numerically stable)

$$\left[\begin{array}{c|cc} E_{11} & 0 & 0 \\ \hline 0 & 0 & E_{22,12} \\ 0 & 0 & 0 \end{array}\right]\begin{bmatrix} \dot{x}_f \\ \dot{x}_{i,1} \\ \dot{x}_{i,2} \end{bmatrix} + \left[\begin{array}{c|cc} A_{11} & A_{12,1} & A_{12,2} \\ \hline 0 & A_{22,11} & A_{22,12} \\ 0 & 0 & A_{22,22} \end{array}\right]\begin{bmatrix} x_f \\ x_{i,1} \\ x_{i,2} \end{bmatrix} + \left[\begin{array}{c} B_1 \\ \hline B_{2,1} \\ B_{2,2} \end{array}\right] u = 0$$

  - after integration (no impulsive input behavior), consistent values are

$$x_f = x_f^- - E_{11}^{-1} A_{12,1} A_{22,11}^{-1} E_{22,12} (x_{i,2} - x_{i,2}^-)$$

$$x_{i,1} = A_{22,11}^{-1}(-B_{2,1}u + E_{22,12}\dot{x}_{i,2}) - A_{22,12}x_{i,2}$$

$$x_{i,2} = -A_{22,22}^{-1} B_{2,2} u$$

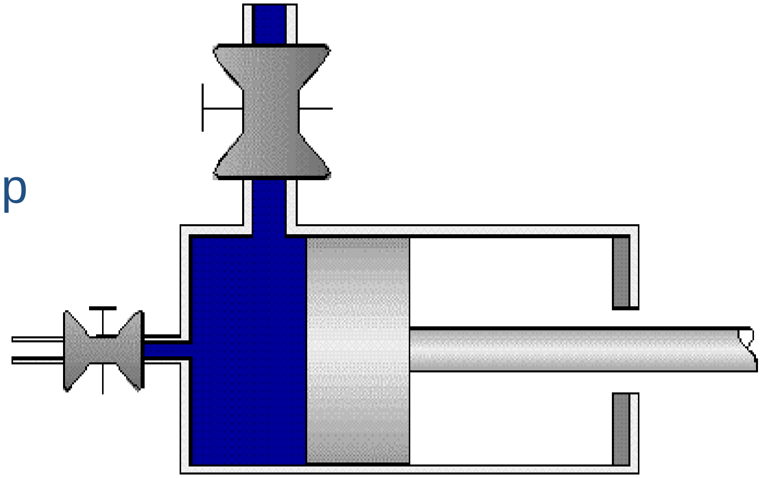
The Hydraulic Actuator

- Generalized State Jumps for Each Mode

Mode	Projection
α_{00}	$f_{rel} = 0$ $v_p = 0$
α_{01}	$v_p = (m_p v_p^- - I_{rel} f_{rel}^-)/(m_{rel} + m_p)$ $f_{rel} = (m_p v_p^- - I_{rel} f_{rel}^-)/(m_{rel} + m_p)$
α_{10}	$v_p = v_p^-$ $f_{rel} = 0$
α_{11}	$v_p = v_p^-$ $f_{rel} = f_{rel}^-$

A Scenario

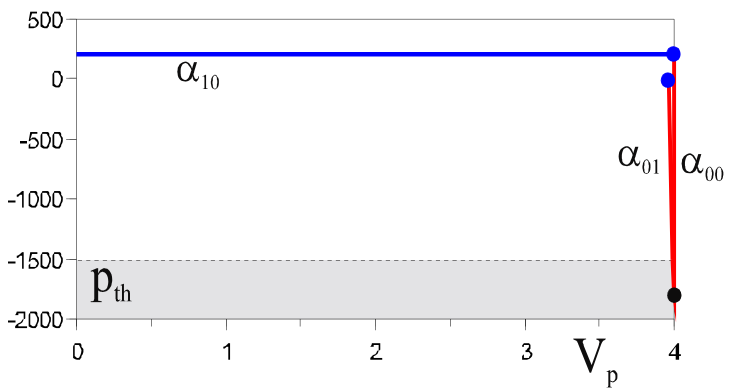
- Intake Valve Is Open
  - piston starts to move
- Intake Valve Closes
  - piston inertia causes pressure build-up
  - pressure reaches critical value
- Relief Valve Opens
  - cylinder pressure decreases



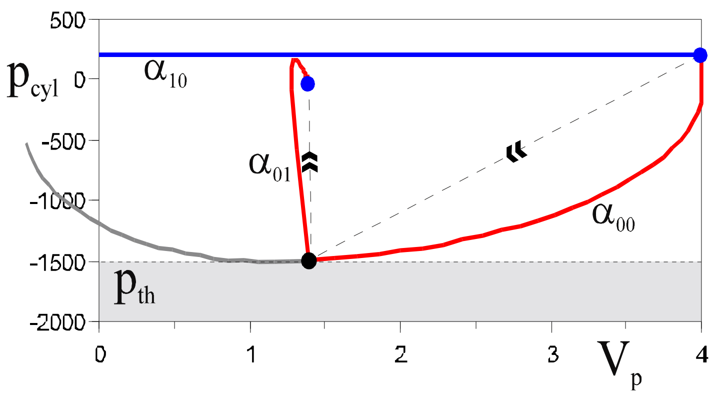
⇒ Interaction Between Mode Transition Behavior

Phase Space of Cylinder Scenario

- Projection Is Aborted
  - immediately
  - after partial completion



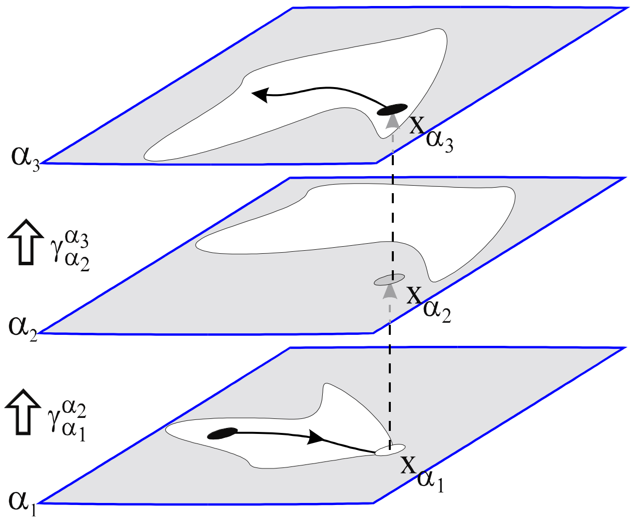
(a)



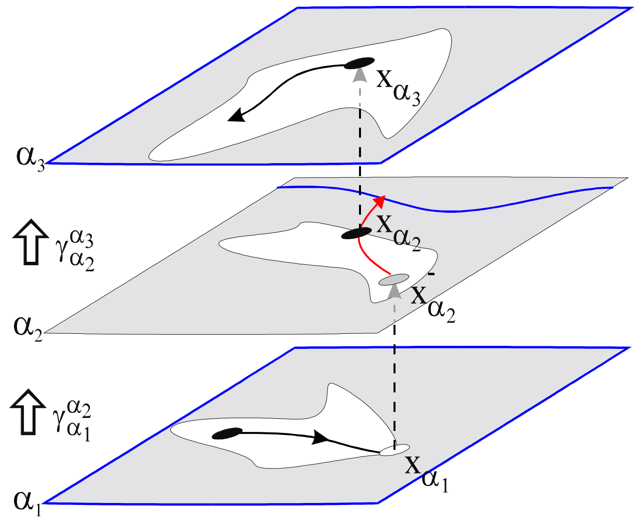
(b)

Sequences of Mode Changes

- (a) State Outside of a Patch in the New Mode
- (b) During Projection State Values are Reached Outside of a Patch in the New Mode



(a)



(b)

Impulses

- High Index Systems May Contain Impulsive Behavior
  - in case of the hydraulic cylinder, $p > p_{th}$, would always hold if not $v_p = v_p^-$
  - unknown where the patch is abandoned
- In-Depth Analysis of Switching Conditions
  - solve for required $x(t)$
  - compute earliest $t = t_s$ at which $\gamma(x(t), u(t), t) \geq 0$
  - substitute t_s to compute $x(t_s)$
- Complex Switching Structure
- Additional Difficulty When Interacting Fast Transients (e.g., collision)

Detailed Analysis of the Projection

- Cylinder Example (Imaginary Eigenvalues, $\lambda = \lambda_r + \boldsymbol{i}\,\lambda_i$)
  - from detailed model
    - solve for $\boldsymbol{p}$

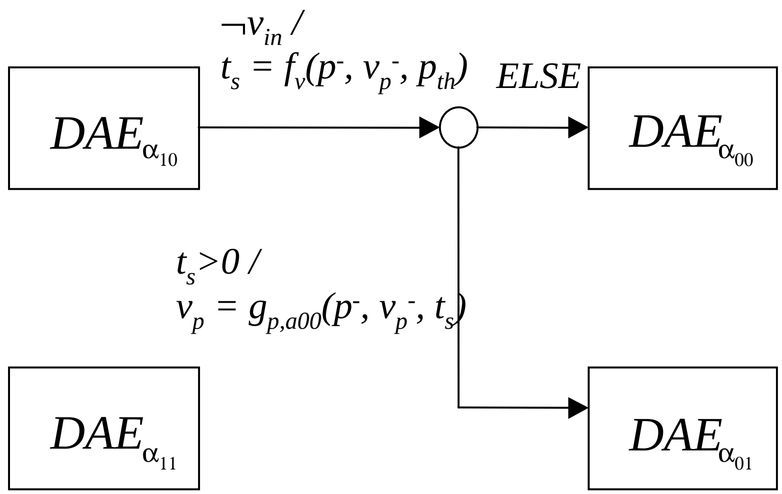
$$p(t) = e^{\lambda_r t}\left(p^- \cos(\lambda_i t) - \frac{1}{\lambda_i}\left(\frac{1}{C_1} v_p^- + \lambda_r p^-\right)\sin(\lambda_i t)\right)$$

    - substitute $\boldsymbol{t}$ at which $\boldsymbol{p(t)} > \boldsymbol{p_{th}}$

$$v_p = e^{\lambda_r t_s}\left(v_p^- \cos(\lambda_i t) - \left(\frac{R_2}{I_1} v_p^- - \frac{p_1}{I_1} + \lambda_r v_p^-\right)\frac{\sin(\lambda_i t_s)}{\lambda_i}\right)$$

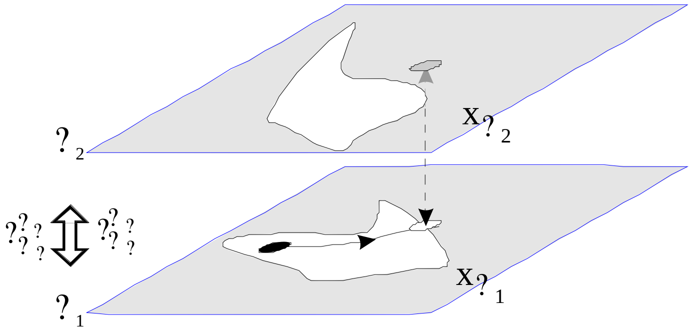
Complex Switching Structure

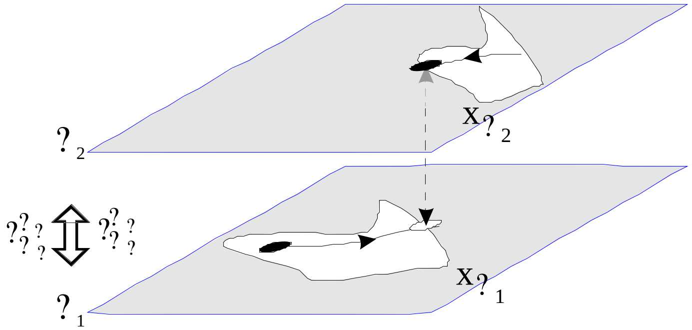
- Explicit Re-Initialization



Chattering

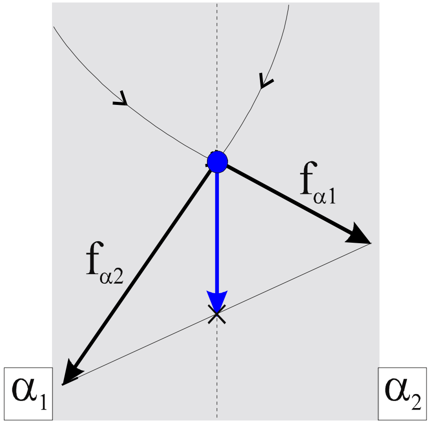
- What If the New Mode Switches Back
  - immediately ⇒ inconsistent model, no solution
  - after infinitesimal period of time ⇒ chattering behavior, solve with
    - equivalent control
    - equivalent dynamics





Equivalent Dynamics

- Chattering
  - fast component
    - remove
  - slow component
    - weighted mean of instantaneous vector fields (Filippov Construction)
  - sliding behavior



Ontology

- Phase Space Transition Behavior Classification
  - mythical (state invariant)
  - pinnacle (state projection aborted)
  - continuous
    - interior (continuous behavior)
    - boundary (further transition after infinitesimal time advance)
    - sliding (repeated transitions after each infinitesimal time advance)
- Combinations of Behavior Classes

Conclusions

- Mode Transition Behavior
  - Rich
  - Complex
- Requires
  - special algorithms/computations
  - model verification analyses
- How to Efficiently Generate Behavior (e.g., for Real-time Applications)?