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A simplified car-following theory: a lower order model

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Abstract

A very simple "car-following" rule is proposed wherein, if an nth vehicle is following an $(n-1)$th vehicle on a homogeneous highway, the time-space trajectory of the nth vehicle is essentially the same as the $(n-1)$th vehicle except for a translation in space and in time. It seems that such a rule is at least as accurate as any of the more elaborate rules of car-following that have been proposed over the last 50 years or so. Actually, the proposed model could be interpreted as a special case of existing models but with fewer parameters and a different logic. At least this should form a reasonable starting point for investigating other phenomena.

Any discussion of the history of car-following theories will be postponed until later. Start from basics. If an nth vehicle is following an $(n-1)$th vehicle (which is following an $(n-2)$th vehicle, etc.), the goal of any model of car following is to describe how the trajectory $x_n(t)$ of the nth vehicle, its position at time t, depends on the trajectory $x_{n-1}(t)$ of the $(n-1)$th vehicle. (This immediately implies that there is no causal connection between the nth vehicle and a following $n+1$th vehicle).

If the $(n-1)$th vehicle were traveling at a constant velocity v

$$x_{n-1}(t) = x_{n-1}(0) + vt$$

the nth vehicle should also travel with an average velocity v, as illustrated in Fig. 1. Otherwise the nth vehicle would either collide with the $(n-1)$th vehicle or would drift far behind. The $(n-1)$th vehicle presumably is following the $(n-2)$th vehicle which is traveling at a velocity v, which, in turn, is following another vehicle. Our model will not deal with the question of what determines the value of v. It is considered as "given", but there must be some vehicle or circumstance which determines its value.

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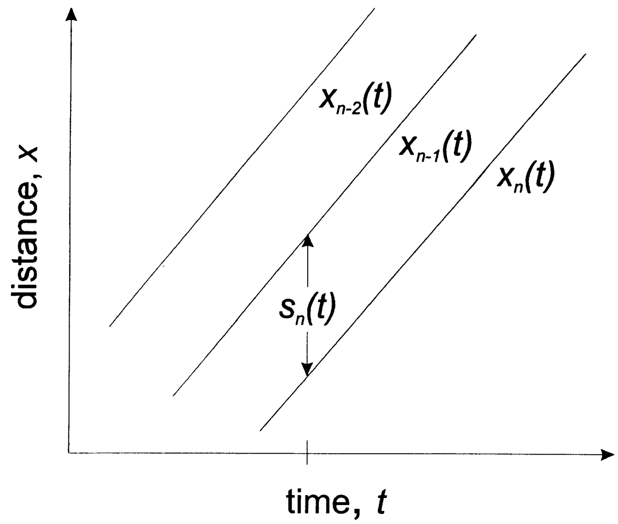

Fig. 1. Constant velocity vehicle trajectories.

The spacing $x_{n-1}(t) - x_n(t)$ between vehicles n and $(n-1)$ at time t could change with time, but, if the highway is homogeneous, it should remain nearly constant at some value s_n. The s_n could be different for different vehicles and it will also depend on the v.

Suppose that for each driver there is some empirical relation between the v and s_n. Certainly if v increases, the drivers will want a larger s_n. This could be idealized by a linear relation as illustrated in Fig. 2 at least over some range of v. Each driver would also have some "desired speed" V_n and if the $(n-1)$th vehicle should travel with $v > V_n$, the nth vehicle would travel at the velocity V_n as illustrated in Fig. 2 by the dashed line, and the two vehicles would separate. The v cannot be negative and the value of s_n for $v = 0$ might be somewhat less than the value provided by the linear relation, as illustrated by the dot for $v = 0$. The value of v could be the V_k for some vehicle downstream $k < n$.

Now suppose that the $(n-1)$th vehicle travels at a nearly constant v for some period of time but then changes velocity, eventually acquiring a nearly constant velocity v'. The actual trajectory may look like the path shown in Fig. 3 by the broken line, but suppose the two constant velocity segments are extrapolated until they intersect, the solid lines, and that the actual trajectory stays close to the solid lines. The nth vehicle will do likewise with the junction of the solid lines for the nth vehicle displaced relative to the $(n-1)$th vehicle by a space displacement d_n and time displacement τ_n.

From the little box of sides d_n and τ_n in Fig. 3 one can see that

$$d_n + v\tau_n = s_n, \qquad d_n + v'\tau_n = s'_n,$$

from which it follows, that if v, s_n and v', s'_n lie on the line of Fig. 2, the slope of this line is τ_n and its intercept at $v = 0$ is d_n.

For the straight line relation between v and s_n as in Fig. 2, the τ_n and d_n are independent of the v or v' of Fig. 3. If the velocity were to change again from v' to v'', the τ_n and d_n would be the same

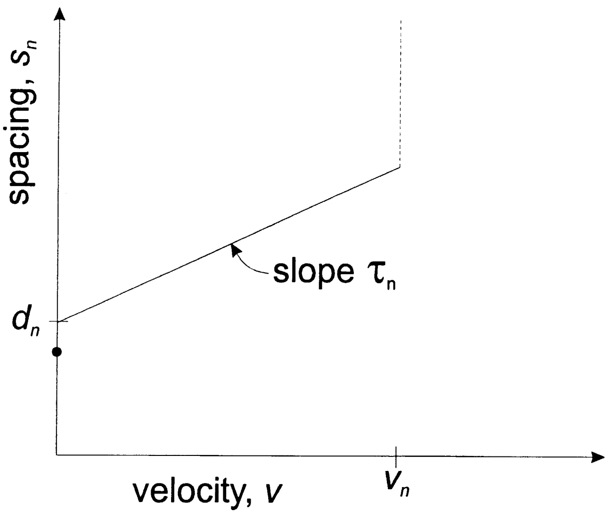

Fig. 2. Relation between spacing and velocity for a single vehicle.

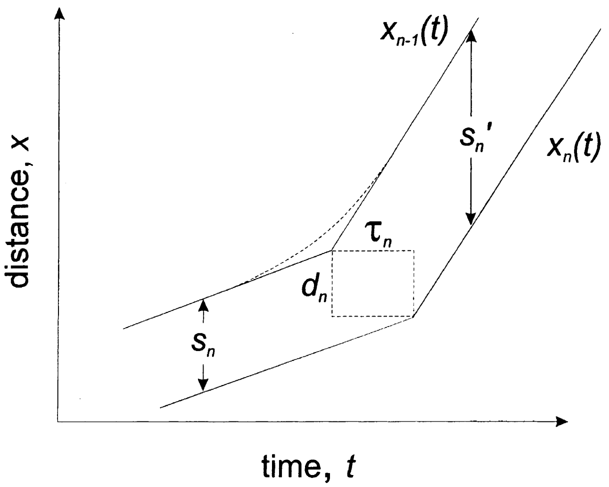

Fig. 3. Picewise linear approximation to vehicle trajectories.

for the second velocity change. Indeed the piecewise linear trajectory $x_n(t)$ would be simply a translation of the piecewise linear $x_{n-1}(t)$ by a distance d_n and a time τ_n, i.e.,

$$x_n(t + \tau_n) = x_{n-1}(t) - d_n. \tag{1}$$

Our proposed model of car-following is that the nth vehicle trajectory will (approximately) follow the trajectory of the $(n-1)$th as in (1) for some appropriate values of τ_n and d_n.

The details of how a following vehicle manages to stay close to relation (1) are not important provided that the driver is capable of doing so. To do so, however, should not be difficult because, if the lead vehicle should increase (decrease) its velocity, the following vehicle does not need to respond immediately. It can wait until the spacing has increased (decreased) to a value comparable with its value for the new velocity. Indeed, the slope τ_n in Fig. 2 derives from what the driver considers to be a safe driving distance which implies an ability for him to respond comfortably to anything the $(n-1)$th vehicle may do.

In the analysis of observations, one would not typically observe the notion of every vehicle, but only some suitable "macroscopic" behavior. It is clear from an iteration of (1) that the τ_n and d_n are the natural parameters (rather than $1/\tau_n$, for example) because (1) implies

$$x_n(t + \tau_n + \tau_{n-1} + \cdots + \tau_1) = x_0(t) - d_n - d_{n-1} - \cdots - d_1 \tag{2}$$

The $x_n(t)$ is a suitable translation of $x_0(t)$.

The τ_n and d_n are expected to vary considerably from one vehicle to the next because some drivers like to follow closely at perhaps a minimum safe driving distance whereas others like to leave a comfortable cushion. It might be reasonable to assume that the (τ_n, d_n) vary as if they were sampled independently from some joint probability distribution with coefficients of variation comparable with 1 (may be 1/2 or 1/3). The velocities of successive vehicles, on the other hand, should show very little (negligible) random variation.

If we let

$$\bar{\tau} = \frac{1}{n}\sum_{k=1}^{n} \tau_k \quad \text{and} \quad \bar{d} = \frac{1}{n}\sum_{k=1}^{n} d_k, \tag{3}$$

the arithmetic average of the τ's and d's, then the average wave speed is $\bar{d}/\bar{\tau}$. The wave, however, propagates like a "random walk" with independent increments vector (τ_k, d_k) in the two-dimensional space of distance–time.

Some people would prefer to describe the macroscopic behavior in terms of flows q and densities k rather than velocities and spacings. A "stationary flow" will be interpreted here as some region in the x, t plane where all vehicles are traveling at nearly the same velocity v, but possibly random spacings and headways.

If

$$s_n = d_n + v\tau_n$$

and all vehicles have the same velocity, then

$$\bar{s} = \bar{d} + v\bar{\tau}$$

The k is interpreted as $1/\bar{s}$ and the v can be interpreted as q/k. Thus

$$q = \frac{1}{\bar{\tau}} - \frac{\bar{d}}{\bar{\tau}}k \tag{4}$$

provided that the v is less than the desired speed V_k of any vehicle. The q is a linearly decreasing function of k with coefficients depending on the means $\bar{\tau}$ and $\bar{d}$.

This establishes the connection between this model and some "fluid models". If, however, for light traffic, the average velocity should exceed the desired speed of some vehicles, but vehicles cannot pass each other, the vehicles would form platoons behind slow vehicles (unless all vehicles had the same V_n). In the usual fluid models it is implied either that all vehicles have the same desired speed or that faster vehicles can pass slower ones. We will be concerned here, however, only with the behavior of vehicles if the velocity is less than V_n.

To see the connection between (1) and some of the more traditional car-following models, suppose that the nth vehicle could follow (1) exactly and the $(n-1)$th vehicle followed some smooth trajectory.

The $x_n(t+\tau_n)$ can be represented as

$$x_n(t+\tau_n) = x_n(t) + \tau_n v_n(t+T_n) \tag{5a}$$

$$\cong x_n(t) + \tau_n v_n(t) + \tau_n T_n a_n(t) \tag{5b}$$

with $v_n(t)$ the velocity of the nth vehicle at time t and $a_n(t)$ its acceleration. The form (5a) follows from the "mean value theorem"of calculus for T_n of some value between 0 and τ_n but, if the function is smooth, the value of τ_n should be approximately

$$T_n = \tau_n/2. \tag{6}$$

Similarly (5b) follows from an expansion of the v_n in (5a), but again the value of T_n should be as in (6).

Substitution of (5a) in (1) gives

$$v_n(t+T_n) = \frac{1}{\tau_n}[x_{n-1}(t) - x_n(t)] - \frac{d_n}{\tau_n} \tag{7}$$

The usual interpretation of this is that the nth driver chooses a velocity based upon the value of the spacing time T_n earlier, but here the T_n is $\tau_n/2$. The derivative of (7) with respect to t gives

$$a_n(t+T_n) = \frac{1}{\tau_n}[v_{n-1}(t) - v_n(t)] \tag{8}$$

which means that the nth driver chooses an acceleration based on the velocity difference a time T_n earlier.

If (5b) is substituted in (1) then

$$T_n a_n(t) = \frac{1}{\tau_n}[x_{n-1}(t) - x_n(t)] - d_n/\tau_n - v_n(t). \tag{9}$$

The usual interpretation of this is that the nth driver chooses an acceleration proportional to his deviation from the equilibrium curve of Fig. 2, with a "relaxation time" T_n.

Eqs. (7)–(9) describe approximately what the driver would have to do to stay exactly on the curve (1). But we do not really care if he does or not as long as he stays close to it. We might, however, be concerned with the possibility that deviations from (1) might accumulate if (1) is iterated over many vehicles as in (2).

This model describes only a part of the story. It does not specify who is the "lead vehicle", if any, or where changes in velocity originate. It only models how changes in velocity would propagate down a line of vehicles on a homogeneous highway.

1. Comparisons with other models and observation

Perhaps the earliest car-following model is that proposed by Herrey and Herrey (1945). They postulated that a driver would maintain a minimum "safe driving distance" but they included in this distance the stopping distance, thus obtaining a spacing which was a quadratic function of the velocity. There was no experimental confirmation of their theory but this paper is significant in that it is one of the very few papers that describes car-following in terms of vehicle trajectories rather than just velocities, spacings, etc. The approach is quite similar to what is done here.

Pipes (1953) assumed a relation as in (7) but with $T_n = 0$. He was concerned mostly with the details of how a disturbance would disperse as it propagates, which is a feature of (7) if T_n is small. There was no experimental observation and indeed such effects do not obviously exist. It seems that the purpose of the paper was to illustrate an application of Laplace transforms.

Kometani and Sasaki (1958) proposed a model similar to (7) but with an unspecified T_n. The T_n was interpreted as a "reaction time". They were mostly concerned with how the value of T_n would influence the stability, whether disturbances would amplify or decay as they propagate.

Kometani and Sasaki (1961) did some car-following experiments and obtained a T_n which was nearly equal to τ_n (rather than $\tau_n/2$), but their observations were done with the lead vehicle undergoing rather violent periodic acceleration and decelerations, and a following vehicle apparently trying to follow as close as he dared. It is doubtful that (1) would apply under such circumstances.

Chandler et al. (1958) proposed a model as in (8) possibly with $T_n \neq \tau_n/2$. They also were concerned mostly with how the value of T_n affected stability. If $T_n < \tau_n/2$ disturbances tend to decay or disperse, but if $T_n > \tau_n/2$ they tend to amplify. They made observations of accelerations and velocity differences under more typical driver behavior than Kometani and Sasaki and did a correlation analysis with various values of the T_n. Different drivers were found to have different values of the T_n and τ_n, but the average value of the T_n/τ_n was found to be nearly 1/2 as in (6). Herman's (1992) interpretation of this was that drivers "are driving on the margin of stability". Indeed this is the logical interpretation if one believes that drivers would behave according to (8) with $T_n \neq \tau_n/2$.

Helly (1961) did a similar analysis with data from two other locations and found that the T_n was reasonably close to $\tau_n/2$, even though the T_n varied considerably among drivers and locations.

That the observed value of T_n is close to $\tau_n/2$ represents only a rather weak confirmation that (1) may be correct because accelerations and velocity differences are quite noisy. Conclusions may be sensitive to how one evaluates averages. The more important question is if (1) can be used to describe coarse scale phenomena.

The logic of Chandler et al. is quite different from that described here. They proposed that, since a driver controls his accelerator pedal or brake, any car-following model should describe how the driver of an nth vehicle controls his acceleration (or deceleration) in response to his observation of the motion of the $(n-1)$th vehicle, but that there would be a time lag in his response.

They did consider the possibility that the nth driver might choose an acceleration as in (9), but with a time lag for the acceleration. They tested the theoretical stability and concluded that (9) was stable for $T_n / \tau_n < 1/2$, independent of any time lag. They did not, however, try to fit (9) to observations. Actually, this would be rather difficult because, with a time lag, (9) would have four adjustable parameters whereas (8) has only two. Equations analogous to (9) have also been used as a basis for some of the "higher order" fluid models.

Eq. (8) is not, by itself, sufficient to describe car following. For a second-order equation one needs two "integration constants"to construct a solution. Whereas in a typical initial value problem one might arbitrarily specify both an initial velocity and a spacing, a first integration of (8) has the form (7) with the integration constant d_n/τ_n. The d_n/τ_n, however, is a property of the driver, not the initial conditions. If in (7), the T_n is an adjustable parameter then (7) has three parameters.

The logic behind (1) is that the nth driver somehow manages to adjust his spacing depending upon the velocity (either his or the $(n-1)$th vehicle's, which are nearly equal). The theory has only two parameters per driver, the τ_n and d_n; the time lag T_n is $\tau_n/2$.

Del Castillo (1996) proposed a model equivalent to

$$v_n(t+T_n) = v_{n-1}(t) \tag{10}$$

which is the time derivative of (1). He gave no behavioral justification of (10). His purpose was to compare this model with the car-following models and the LWR theory. He concluded that for (10) to be consistent with (7) or (8) the T_n would need to be $\tau_n/2$, but he failed to observe that (10) integrates to (1) with the integration constant d_n.

That the best fit value of T_n in (8) turns out to be close to $\tau_n/2$ is undoubtedly no coincidence. It would suggest that the T_n has nothing to do with the reaction time of the driver except in so far as the safe driving distance and therefore the T_n is related to the driver's reaction time.

2. Non-linear effects

An obvious generalization of (7) would be to choose some non-linear function of the spacing on the right-hand side or its corresponding derivative in (8). Apparently the original motivation for doing this (Gazis et al., 1961), was to conform with some non-linear relation between stationary values of q and k (or v and s, etc.).

There seems to be no end to papers attempting to fit curves to observed values of (q, k), or other pairs of "macroscopic" variables. Most of these observations of (q, k) are made from counts of vehicles during prespecified time intervals (1 or 5 min). For densities above that for maximum flow (where car-following may exist), these tend to show a wide scatter, as would be expected if the τ_n, d_n are independent random variables. But if larger time intervals are chosen, the velocity or flow may not be stationary during the time interval. Some people seem to believe that the flow is "unstable" or "chaotic" over some range of densities near that of maximum flow because the observed (q, k) points seem to be widely scattered and follow crazy paths. One could make a least square fit to these data with many different shaped curves.

Del Castillo and Benitez (1995), on the other hand, used various statistical tests to extract from time series data periods of time during which the flow or average velocity was nearly stationary. Using data only from these time periods, they could establish a well-defined relation between velocity and spacing. The data extended from velocities as low as 20 km/h nearly to the free-flow average of about 110 km/h. To fit curves to these data over the whole range of velocities they needed to use some nonlinear curves, but between 20 km/h and about 100 km/h the relation was very nearly linear.

Cassidy (1998) has also shown that if one uses only data from time intervals (as long as possible) during which the flow seems to be nearly stationary, that one can obtain a well-defined relation between flow and occupancy. He extracted these time intervals from curves of cumulative flow vs. time. His data are only over a rather limited range of occupancies mostly near that of maximum flow, but, for occupancies above this maximum flow, one could easily fit a linear relation. (This would imply a linear relation between q and k).

It is certainly not obvious from the observation of (v, s) or (q, k) that one needs to assume a non-linear relation.

One of the consequences of a non-linear relation between q and k, or v and s, is that the wave velocity is different for different values of q or k. This would cause waves for different flows to either diverge or focus. This was discussed in great detail by Lighthill and Whitham (1955) and Richards (1956), LWR, for a fluid model. In the fluid theory focusing waves form shocks (discontinuities) whereas diverging waves tend to disperse changes in flow. If the q vs k relation is concave, deceleration waves tend to focus, acceleration waves to diverge.

Some people reject the LWR theory because traffic should not show discontinuities (infinite decelerations). The discontinuities, however, are a result of treating traffic as a continuous fluid. A non-linear version of (7) would portray a shock as a change in velocity over a distance comparable with the vehicle spacing (Newell, 1961), which propagates with no change in shape.

The theoretical consequences of a non-linear relation are well understood but the evidence that non-linear effects exist in real traffic is highly questionable. Foster (1962) tried to verify the LWR theory for vehicles starting from a traffic signal. He tried to fit a non-linear curve to values of (q, k), but it appears that a linear curve would fit just as well. He also tried to verify that the acceleration waves diverge, but in his illustration of vehicle trajectories, it appears that he could just as well have concluded that all waves have nearly the same velocity.

Herman and Rothery (1965) observed the motion of an 11-vehicle platoon with the lead vehicle making various maneuvers. They found no obvious dependence of the wave velocity on the vehicle velocity, although with only 11 vehicles, there was considerable random fluctuation. They did observe some difference in the wave velocity for acceleration and deceleration and they also noted that vehicles have some difficulty following at low velocities (less than 20 km/h).

The most convincing evidence comes from recent and ongoing observations. There are now several places in the world where vehicle detectors have been installed in every lane of a freeway at various spacings. These automatically record flow, occupancy, and sometimes velocities at each detector location. There are also some places where video cameras have been installed.

Kerner and Rehborn (1996) have tracked the propagation of two "traffic jams", temporary stoppages, through a 13 km section of highway in Germany for about 50 min. The jam propagates with virtually no change in shape with a rather sudden drop in flow and a sudden recovery. This could be true only if each vehicle was following the vehicle ahead of it with nearly the same trajectory except for a translation in space and time and that this would be valid even after an iteration over thousands of vehicles! There is no difference between the wave velocity for acceleration and deceleration and no spreading of either the acceleration or deceleration waves. Unfortunately, the jams apparently originated outside the boundaries of observations so there was no explanation as to why the jams occurred.

Windover (1998) tracked the propagation of about 60 flow changes over a 2 km section of I880 in Hayward California. He verified that disturbances travel like a random walk with predictable statistical variation. Within the statistical uncertainty of the random walk, he found no observable dependence of the wave velocity on flow or whether the wave was an acceleration or deceleration. There was no tendency for waves to focus or diverge. The flow changes included some complete stoppage but all changes seemed to originate outside the region of observation. The observations were between on or off ramps.

The conclusion is that there seems to be little, if any evidence of these non-linear effects except possibly for velocities close to the desired velocities, flows close to the maximum, or for very low velocities.

3. Applications

The present model applies only to a homogeneous highway with negligible passing, but even then it does not specify who is the lead vehicle. To apply the model, one must supplement this with some boundary conditions or some specification of what happens at some inhomogeneity in the highway.

An obvious application of the model would be to the behavior of a long queue of vehicles passing through a signalized intersection. The first driver to be stopped at the intersection when the signal turns red is not following the vehicle ahead; he is responding to the signal. He decelerates at a finite rate and comes to a stop. The next driver is following the first but he also can see the traffic signal. He may start his deceleration earlier than predicted by (1), in response to the signals, and decelerate more gradually.

The LWR theory, which does not deal with individual vehicles, predicts that there should be a deceleration shock. But (1) would predict that the jth vehicle decelerates no faster than the first. Actually he probably decelerates at a slower rate.

When the signal turns green, the first driver accelerates presumably until he reaches his desired velocity. The second driver, if he follows (1), would accelerate at the same rate but delayed by a time τ. Actually he may accelerate less if the first driver accelerates faster than the second driver would like. If the second driver chooses not to keep up with the first, he may become the "lead vehicle". The LWR theory disregards the time to accelerate and lets the lead vehicle, in effect, immediately to assume the free flow velocity.

In any case the traffic signal can be interpreted as an inhomogeneity in the highway and one must supplement (1) with some other rules of how vehicles react to the inhomogeneity.

A fluid theory analogous to the LWR theory but with a triangular relation between q and k has been used to describe queue propagation on freeways (Newell, 1993). There it is assumed that each point on the freeway has a specified maximum flow (capacity). On or off ramps are idealized as point junctions. It is also assumed that flows entering or leaving can merge smoothly and that the flow is constrained by what can pass some bottleneck downstream. Although this fluid theory is consistent with (1) over homogeneous sections of freeway between ramps or bottlenecks, it also implies some rules of behavior at the junctions. These rules involve little more than a conservation of vehicles, but it is assumed that there is a well-defined capacity and that vehicles can change lanes, merge, or do whatever is necessary to accommodate entering or exiting vehicles.

If there is some failure in the theory it is probably at or near these inhomogeneities. One cannot, however, model the behavior at inhomogeneities without some observations of what actually happens.

4. Conclusion

A simple model of car following is proposed for a homogeneous highway in which an nth vehicle follows the same trajectory as the $(n-1)$th vehicle except for a translation in space and time. The model is consistent with the model and observations of Chandler et al. but the present model has no driver "reaction time". It is consistent with the LWR theory with a triangular shaped $q-k$ curve but it has no shocks or diverging waves (for which there is little observational evidence), except for a possible interface between freely flowing and congested traffic (the back of a queue).

To apply the theory one must specify what happens at some points downstream presumably at some inhomogeneity in the highway. Other models of traffic flow are similarly incomplete or, if they attempt to describe the behavior at inhomogeneities, the model may not be consistent with what actually happens. If the present model is correct, future attention can be focused on where and why flow changes originate.

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